COMPLEX HEXAPOLAR NUMBERS AND THE COMPLEX HEXAPOLAR PLANE: AN OVERVIEW *DRAFT*

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1. Introduction and Motivation

The complex hexapolar numbers ($\mathbb{Y}_{C_6}$) are an instance of a complex multipolar number system ($\mathbb{Y}_{C_p}$) that extends the complex numbers ($\mathbb{C}$). (A more formal treatment of their construction appears elsewhere[2]; the purpose of this document is generally expository and for quick reference.) The complex hexapolars are the minimum nontrivial multipolar embedding of the linear continuum of the reals ($\mathbb{R}$) and its extension by an imaginary fourth root of unity (i); $\mathbb{R} \in \mathbb{C} \in \mathbb{Y}_{C_p} \iff (p/2)\text{mod}2 = 1$ and $p \geq 6$. Whereas quaternions ($\mathbb{H}$) extend and embed $\mathbb{C}$ such that multiplication is not necessarily commutative for $a \in \mathbb{H}$ such that $a \notin \mathbb{C}$, multipolar addition is not necessarily associative for expressions containing more than two unlike signs ($\mathbb{Y}_{C_6}$ contains a total of six signs, four of which $\notin \mathbb{C}$). For $\mathbb{Y}_{C_p}$, which may be represented as an ordered pair of multipolar numbers, corresponding components of unlike sign and equal magnitude annihilate under a generalized addition operation over which multiplication distributes. Multiplication of real-like or imaginary-like multipolar components necessarily associates and multiplication of complex multipolars is not necessarily associative; all nonzero numbers in $\mathbb{Y}_{C_6}$ are contained by at least one of three intersecting subsets ($\mathbf{L}_0$, $\mathbf{L}_1$, $\mathbf{L}_2$) that each form a group under multiplication (i.e., elements of those subsets associate and have inverses) such that $\mathbb{C} \in \mathbf{L}_0$ and $\mathbb{C} \neq \mathbf{L}_0$.

2. How the complex hexapolars relate to complex ($\mathbb{C}$) and real ($\mathbb{R}$) numbers

For $i \in \mathbb{C}$ and $s \in \mathbb{Y}_{C_6}$, $s^3 = i$ such that $s \notin \mathbb{C}$; $s \neq -i = s^9$. $s^{12} = 1, s^6 = -1$. In general, for s^n such that $n \in \mathbb{N}$ and $n\text{mod}2 = 1$, s^n is an additive inverse of an imaginary or imaginary-like element whereas for s^m such that $m \in \mathbb{N}$ and $m\text{mod}2 = 0$, s^m is an additive inverse of a real or real-like element.

TABLE 1. Conversion between elements of $\mathbb{R}, \mathbb{C}$, and $\mathbb{Y}_{C_6}$

Number System													
$\mathbb{R}$	0	-	-	-	-	-	-1	-	-	-	-	-	1
$\mathbb{C}$	0	-	-	i	-	-	-1	-	-	$-i$	-	-	1
$\mathbb{C}$ (as i^n)	0	-	-	i	-	-	i^2	-	-	i^3	-	-	i^4
$\mathbb{Y}_{C_6}$	0	s	S1	Ss	T1	Ts	-1	$-s$	P1	Ps	H1	Hs	1
$\mathbb{Y}_{C_6}$ (as s^n)	0	s^1	s^2	s^3	s^4	s^5	s^6	s^7	s^8	s^9	s^{10}	s^{11}	s^{12}

TABLE 2. Sign multiplication table of $\mathbb{Y}_{C_6}$

	+1	**S1**	**T1**	**-1**	**P1**	**H1**
+1	+1	S1	T1	-1	P1	H1
S1	S1	T1	-1	P1	H1	+1
T1	T1	-1	P1	H1	+1	S1
-1	-1	P1	H1	+1	S1	T1
P1	P1	H1	+1	S1	T1	-1
H1	H1	+1	S1	T1	-1	P1

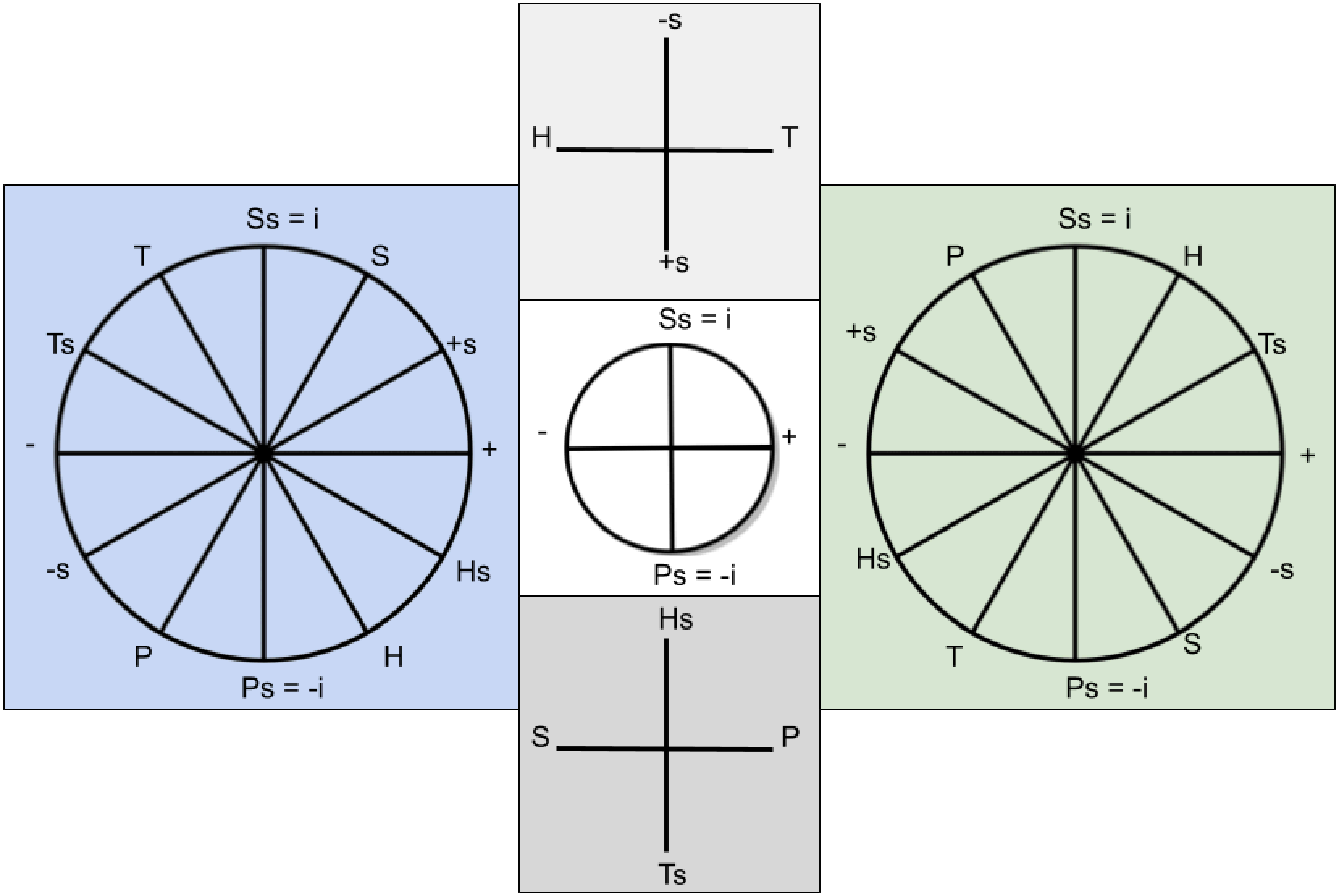
3. What is the complex hexapolar plane? What does that even mean?

As the complex plane is a geometric interpretation of the complex number system, the complex hexapolar plane is a geometric interpretation of the complex hexapolars. The complex hexapolar plane embeds the complex plane; one may also say that it is the complex plane extended by other elements. Whereas quaternions extend the complex plane with two imaginary basis vectors that introduce an isometry with a four-dimensional vector space over the real numbers, complex hexapolars contain the same quantity of basis vectors as the complex numbers (two: corresponding components of unlike sign are linearly dependent), and so one speaks in terms of a two-dimensional plane. The imaginary axis is extended by four classes of elements that are inverses to each other, i, and $-i$ under a generalized addition; this implies a real axis extended by four classes of elements that are inverses to each other, 1, and -1 under a generalized addition.

Whereas the complex plane contains two axes each containing two mutually inverse directions of extension π radians apart, each axis of the complex hexapolar plane contains a total of six mutually inverse directions of extension that may be computed π radians apart using a generalized inner product operation. The complex plane, with its two signs and two axes (2^2), contains four quadrants (or, more generally, four orthants) and the complex hexapolar plane, with its six signs and two axes (6^2), contains 36 orthants, four of which belong to the complex plane.

3.1. **36 orthants that include the usual four quadrants? How does that work?** A heuristic for building intuitions regarding the structure of the complex hexapolar plane might be to think of it as the complex plane extended by two sets of metaphorically twinned regions that all share an origin such that $\mathbb{Y}_{C_6}$ could be said to consist of:

Visualization of the complex hexapolar plane
by Ben Blohowiak



- $\mathbf{L}_0$, which contains the complex plane (the four quadrants/orthants $\in \mathbb{C}$) and eight orthants which may be characterized as a pair of isometric four-quadrant analogues of the complex plane such that they contain each other's inverses under multiplication–and each other's square roots–whose points may not be expressed in polar form (e.g., $\neq (s^x)^{2\theta/\pi}$ and $\notin \mathbb{C}$).
- 24 so-called *long-cycle orthants* (which may be characterized as a pair of isometric 12-orthant regions, $\mathbf{L}_1$ and $\mathbf{L}_2$) whose points are $\notin \mathbb{C}$ and may be expressed in polar form

4. How could someone enhance their intuitions regarding the structure of (or visualize) the complex hexapolar plane?

Many approaches are possible; some involve representing a point on an axis, such as the origin, in more than one location on a map, chart, or depiction.

One representation allocates regions in accord with the subsets $\mathbf{L}_0$, $\mathbf{L}_1$, and $\mathbf{L}_2$ such that the 36 orthants of the complex hexapolar plane is 5 squares: one set of three squares containing four orthants each and a set of two squares containing twelve orthants each. The set of three squares ($\mathbf{L}_0$) contains one corresponding to the complex plane; the other two of that set correspond to regions of the complex hexapolar plane that may be arrived at from a non-origin point in the complex plane via scalar multiplication by s^4 or s^8 that have no polar expression. In total, those three squares contain 12 orthants. The remaining two squares each correspond to 12 orthants of the complex hexapolar plane ($\mathbf{L}_1$, $\mathbf{L}_2$) and may be scaled proportionally (i.e., the area of one square of the set of two squares as equal in total area to the set of three squares). Representing 12 orthants in the four quadrants of each of the set of two squares may proceed according to the mappings $s^{2\theta/\pi} \mapsto i^{2\theta/3\pi}$ and $(s^5)^{2\theta/\pi} \mapsto i^{2\theta/3\pi}$, respectively. In such mappings, axial directions away from the origins of each of the squares in the set of two squares resemble the cutting lines of a 12-slice pie, with axes mapped such that they project as alternating real and imaginary "spokes" extending as if from the central hub of the origin (see attached visualization).

Another way to represent the 36 orthants of the complex hexapolar plane is with a set of nine squares such that each square contains four quadrants (e.g., see table of equivalence classes of four-orthant sets or symmetries of generalized Mandelbrot and Tricorn functions). The combinatorics are straightforward: chunk the 36 orthants into nine four-quadrant sets ("squares") so that they resemble nine versions of $\mathbb{R}^2$. One such four-quadrant square corresponds to the complex plane..

5. Anything noteworthy regarding the unit circle of $\mathbb{Y}_{C_6}$?

The unit circle of the complex plane is embedded in the complex hexapolars ($i^{2\theta/\pi} = (s^3)^{2\theta/\pi}$; $(-i)^{2\theta/\pi} = (s^9)^{2\theta/\pi}$). What may be of interest is that if a unit circle of the complex plane is defined as the set A of elements $a \in \mathbb{C}$ such that $|a| = 1$ and the unit circle of the complex hexapolar plane is defined as the set B of elements $b \in \mathbb{Y}_{C_6}$ such that $|b| = 1$, then $A \in B$ but $A \neq B$.

Unlike points in the complex plane alone, not all nonzero points in the complex hexapolar plane may be expressed in polar form. If the unit circle of the complex hexapolar plane is defined as the set D of elements that are outputs of functions of θ that exponentiate an imaginary-like unit as suggested above, $D \cap B$ but $D \neq B$; the set B contains elements of eight orthants whose points $\notin D$. The outputs of

TABLE 3. Equivalence Classes of 4-Orthant Sets (orthants indicated by signed unit pairs)

	$f(\theta)$	(a,b)	$i(a,b)$	$-1(a,b)$	$(-i)(a,b)$
$\in \mathbb{C}$	$(s^3)^{2\theta/\pi}$	(+1,P1)	(+1,S1)	(−1,S1)	(−1,P1)
$\notin \mathbb{C}$	-	(T1,+1)	(T1,−1)	(H1,−1)	(H1,+1)
$\notin \mathbb{C}$	-	(P1,T1)	(P1,H1)	(S1,H1)	(S1,T1)
$\notin \mathbb{C}$	$s^{2\theta/\pi}$	(+1,+1)	(T1,S1)	(−1,−1)	(H1,P1)
$\notin \mathbb{C}$	$s^{2\theta/\pi}$	(T1,T1)	(P1,−1)	(H1,H1)	(S1,+1)
$\notin \mathbb{C}$	$s^{2\theta/\pi}$	(P1,P1)	(+1,H1)	(S1,S1)	(−1,T1)
$\notin \mathbb{C}$	$(s^5)^{2\theta/\pi}$	(+1,T1)	(P1,S1)	(−1,H1)	(S1,P1)
$\notin \mathbb{C}$	$(s^5)^{2\theta/\pi}$	(P1,+1)	(T1,H1)	(S1,−1)	(H1,T1)
$\notin \mathbb{C}$	$(s^5)^{2\theta/\pi}$	(T1,P1)	(+1,−1)	(H1,S1)	(−1,+1)

such functions of θ trace three distinct paths through subsets of the plane, one of which traverses the four quadrants of the complex plane such that the other two paths pass through twelve non-overlapping orthants (i.e., a set of twelve *long-cycle orthants*) each. In total, cycles of imaginary exponentiation pass through 28 orthants, bypassing 8 remaining orthants.

TABLE 4. $f(\theta)$: Exponentiation of s (orthants indicated by signed unit pairs)

	θ **Range**	$s^{2\theta/\pi}$	$(s^5)^{2\theta/\pi}$	$i^{2\theta/\pi}$	$(-i)^{2\theta/\pi}$
1	$(0,\pi/2)$	(+1,+1)	(+1,T1)	(+1,S1)	(+1,P1)
2	$(\pi/2,\pi)$	(S1,+1)	(H1,T1)	(−1,S1)	(−1,P1)
3	$(\pi,3\pi/2)$	(S1,S1)	(H1,S1)	(−1,P1)	(−1,S1)
4	$(3\pi/2,2\pi)$	(T1,S1)	(P1,S1)	(+1,P1)	(+1,S1)
5	$(2\pi,5\pi/2)$	(T1,T1)	(P1,+1)	(+1,S1)	(+1,P1)
6	$(5\pi/2,3\pi)$	(−1,T1)	(−1,+1)	(−1,S1)	(−1,P1)
7	$(3\pi,7\pi/2)$	(−1,−1)	(−1,H1)	(−1,P1)	(−1,S1)
8	$(7\pi/2,4\pi)$	(P1,−1)	(T1,H1)	(+1,P1)	(+1,S1)
9	$(4\pi,9\pi/2)$	(P1,P1)	(T1,P1)	(+1,S1)	(+1,P1)
10	$(9\pi/2,5\pi)$	(H1,P1)	(S1,P1)	(−1,S1)	(−1,P1)
11	$(5\pi,11\pi/2)$	(H1,H1)	(S1,−1)	(−1,P1)	(−1,S1)
12	$(11\pi/2,6\pi)$	(+1,H1)	(+1,−1)	(+1,P1)	(+1,S1)

5.1. **In the Exponentiation of s table, why only s and s^5 raised to the $2\theta/\pi$, what about the other powers of s?** s^{11} and s^7 trace the same paths as s and s^5, respectively, but in an opposing direction–analogous to $i^{2\theta/\pi}$ versus $(-i)^{2\theta/\pi}$. Powers of s that are evenly divisible by two behave differently when raised to $2\theta/\pi$ than do odd powers (e.g., $s^{12}=1, s^6=-1$). $s^3=i$ and $s^9=-i$; $(s^3)^{2\theta/\pi}$ and $(s^9)^{2\theta/\pi}$ trace in opposing directions the unit circle of the complex plane embedded in the complex hexapolar plane.

6. You mentioned exponentiation; is that defined as the sum of a power series? By repeated multiplication?

It is possible to define an $\exp(z)$ function as the sum of a power series by keeping a running tally of consolidation with each subsequent term in the series; doing so demonstrates the embedding of the complex numbers in the complex hexapolars as the outputs of an exp() function so defined, given complex inputs (e.g., $\mathrm{s}^3\theta$), equal those of the conventional complex exponential. Complex hexapolars $\notin \mathbb{C}$ input to a generalized exp() do converge, or tend toward limits. However, in most cases $\exp(\mathrm{s}\theta){\neq}\mathrm{s}^{2\theta/\pi}$ such that the latter expression may correspond to repeated multiplication and not necessarily the sum of a power series. Whereas the Euclidean magnitude of $\mathrm{s}^{2\theta/\pi} = 1$, the Euclidean magnitude and natural modulus of $\exp(s\theta)$ varies in a self-similar, not-precisely-periodic manner suggestive of fractals and chaos theory; more statistical analysis of the generalized exp() function, both in terms of its magnitudes and component sign combinations, is a direction for future research.

7. Do the properties of complex numbers in relation to their conjugates generalize to the other complex hexapolars $\notin \mathbb{C}$?

Some but not all properties of complex numbers in relation to their conjugates generalize without condition; some complex hexapolars exhibit novel properties in relation to their conjugates not expressed by conjugacy relations $\in \mathbb{C}$ as complex hexapolar conjugation is not necessarily distributive over addition or multiplication. Adding a given complex hexapolar with its conjugate annihilates any imaginary-like part and doubles the magnitude of any real-like part. Similarly, the geometric characterization of a number's conjugate as a reflection across the real-like axis also holds. However, if expressing complex hexapolars in polar form with $\theta, x \in \mathbb{R}_{\geq 0}$, the conjugate of $(s^x)^{2\theta/\pi}$ is not necessarily $(s^x)^{-2\theta/\pi}$, though that method of conjugation holds if $x = 3$ or $x = 9$, as such numbers are in the complex plane ($\mathbb{C} \in \mathbb{Y}_{C_6}$).

The product of a complex hexapolar with its conjugate is a real-like number (i.e., not an imaginary-like number). That product may be positive, nonpositive, or zero. Nonzero conjugate factors that annihilate under multiplication may contain components of unbounded magnitude and may be characterized by a pattern between their components such that a complex hexapolar (a, b) that will annihilate under multiplication with its conjugate may be expressed such that r is a positive real number, x, y are even natural numbers such that $x\text{mod}2 = 0$,y$mod2$=0, $a = r(s^x), b = r(s^y)$, and $(x-4)\text{mod}12 \neq y$ or $(x+2)\text{mod}12 \neq y$. This implies that for a given pair of values x and $x+6$, there is a pair of values y and $y+6$ that will not result in annihilation if (a, b) is multiplied by its conjugate; rather, its magnitude will be squared (such that it equals $2r^2$). In such cases, the coefficient r of s^y may be substituted for a different positive real number r' such that the product magnitude equals $r^2 + r'^2$; such a substitution if $(x-4)\text{mod}12 \neq y$ or $(x+2)\text{mod}12 \neq y$ would result in a product neither zero nor of squared magnitude but rather of magnitude $|r^2 - r'^2|$.

7.1. **You referenced magnitude or distance from the origin; how might someone compute the magnitude of a complex hexapolar number or its analogue in a generalized inner product space?** Previously–in another

TABLE 5. Example products of complex hexapolars with their conjugates

	Polar form $f(r,\theta)$	$a = r(s^x)$	$b = r(s^y)$	$(a,b)(a,b)^*$	$\|\|$ if $b/s^y \neq r$
$\in \mathbb{C}$	$((2r^2)^{1/2})(s^3)^{2\theta/\pi}$	rs^0 or rs^6	rs^2 or rs^8	$(\|(a,b)\|^2)(\mathrm{sgn}(a)^2)$	$r^2 + r'^2$
$\notin \mathbb{C}$	-	rs^2 or rs^8	rs^{10} or rs^4	$(\|(a,b)\|^2)(\mathrm{sgn}(a)^2)$	$r^2 + r'^2$
$\notin \mathbb{C}$	-	rs^4 or rs^{10}	rs^0 or rs^6	$(\|(a,b)\|^2)(\mathrm{sgn}(a)^2)$	$r^2 + r'^2$
$\notin \mathbb{C}$	$((2r^2)^{1/2})s^{2\theta/\pi}$	rs^0	rs^0 or rs^{10}	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})s^{2\theta/\pi}$	rs^2	rs^0 or rs^2	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})s^{2\theta/\pi}$	rs^4	rs^2 or rs^4	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})s^{2\theta/\pi}$	rs^6	rs^4 or rs^6	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})s^{2\theta/\pi}$	rs^8	rs^6 or rs^8	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})s^{2\theta/\pi}$	rs^{10}	rs^8 or rs^{10}	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})(s^5)^{2\theta/\pi}$	rs^0	rs^4 or rs^6	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})(s^5)^{2\theta/\pi}$	rs^2	rs^6 or rs^8	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})(s^5)^{2\theta/\pi}$	rs^4	rs^8 or rs^{10}	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})(s^5)^{2\theta/\pi}$	rs^6	rs^{10} or rs^0	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})(s^5)^{2\theta/\pi}$	rs^8	rs^0 or rs^2	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})(s^5)^{2\theta/\pi}$	rs^{10}	rs^2 or rs^4	0	$\|r^2 - r'^2\|$

piece[2]–, I defined a generalization of an inner product operation for multipolar numbers that preserves the intuitive notion that, say, $1+4s$ and 1+S4s are equidistant from the origin, as one might expect from a Euclidean metric. However, that generalization of inner product may apply to complex multipolar number systems that do not embed the complex plane ($\mathbb{C}$) and those that do alike.

It is conventional (or even 'natural'[1]) to define an inner product space over $\mathbb{C}$ such that the inner product of two complex numbers may be expressed as the product of one multiplied by the conjugate of the other; that naturalistic method of generalizing the inner product operation to complex hexapolars does not necessarily lead to preservation of the intuitive notion that $1 + 4s$ and 1+S4s are equidistant from the origin. Rather, a generalized inner product based in conjugacy relations defines subset regions of the complex hexapolar plane whose points' distances from the origin resemble those of a space under a Euclidean metric (such as $\mathbb{C}$ and the other eight orthants $\in \mathbf{L}_0$) and regions ($\mathbf{L}_1$ and $\mathbf{L}_2$) whose points' distances from the origin resemble those of a space under a Minkowski metric or split-complex plane, including points that are outputs of the functions of θ described above such that points of a "unit circle" so defined may vary in distance from the origin like conjugate unit hyperbolae. Unlike in the case of the split-complex numbers[4], however, each nonzero complex hexapolar has a multiplicative inverse.

8. Wait, a nonzero complex hexapolar number with a null modulus is invertible!?! Does this have something to do with the subsets $\in \mathbb{Y}_{C_6}$ that do not form groups under multiplication (because multiplication of complex multipolars doesn't necessarily associate)?

Let $\mathbf{K}$ be the set that contains purely real-like or imaginary-like hexapolars (i.e., hexapolars having only one nonzero component). Multiplication of such numbers in $\mathbf{K}$ necessarily associates and multiplication of complex, two-component hexapolars does not necessarily associate. All nonzero hexapolars $\in \mathbb{Y}_{C_6}$ are contained in subsets that form groups under multiplication (i.e., elements of those subsets associate and have inverses) and the intersection of those subsets $\mathbf{L}_0, \mathbf{L}_1, \mathbf{L}_2$ is $\mathbf{K}$. ($\mathbb{C} \in \mathbf{L}_0$ and $\mathbb{C} \cap \mathbf{K}$).

For nonzero $a \in \mathbf{L}_0$, the magnitude of $||a|| > 0$. For nonzero $b \in \mathbf{L}_1$, the magnitude of $||b|| \geq 0$ and $b^{-1} \in \mathbf{L}_1$; for nonzero $c \in \mathbf{L}_2$, the magnitude of $||c|| \geq 0$ and $c^{-1} \in \mathbf{L}_2$. For complex (i.e., two-component) $a, a^* \in \mathbf{L}_0$, complex $b \in \mathbf{L}_1$ such that $b^* \in \mathbf{L}_2$ and complex $c \in \mathbf{L}_2$ such that $c^* \in \mathbf{L}_1$. Multiplication of complex $b \in \mathbf{L}_1$ with complex $c \in \mathbf{L}_2$ (or complex $b^* \in \mathbf{L}_2$) does not necessarily associate. In this way, a nonzero complex hexapolar number with a null modulus has a multiplicative inverse even though a split-complex number with a null modulus has no multiplicative inverse.

9. So with one generalized inner product operation, different regions of the complex hexapolar plane may fall under a Euclidean or Minkowski/hyperbolic norm? How do those disparate regions play together when two complex hexapolar planes are orthogonally juxtaposed, as in the case of $\mathbb{Y}^2_{C_6}$?

Similar to how one may associate $\mathbb{C}$ with $\mathbb{R}^2$ and likewise regions of $\mathbb{Y}_{C_6}$ with $\mathbb{R}^2$ or $\mathbb{R}^{1,1}$, $\mathbb{Y}^2_{C_6}$ embeds analogues of such planar subspaces as well as spaces associated with $\mathbb{R}^4$, $\mathbb{R}^{1,3}$, and $\mathbb{R}^{2,2}$. The combinatorics of such associations is expressed in the table below. As indicated above, $\mathbb{C}$ and the other orthants $\in \mathbf{L}_0$ and the long-cycle orthants $\in \mathbf{L}_1$ and $\in \mathbf{L}_2$.

Table 6. $\mathbb{Y}^2_{C_6}$ Comprised by Planes A & B: Associations w/Subspaces

	$\mathbf{L}_0$**(A)**	$\mathbf{L}_1$ **or** $\mathbf{L}_2$ **(A)**
$\mathbf{L}_0$ **(B)**	$\mathbb{R}^4$	$\mathbb{R}^{1,3}$
$\mathbf{L}_1$ **or** $\mathbf{L}_2$ **(B)**	$\mathbb{R}^{1,3}$	$\mathbb{R}^{2,2}$

10. $\mathbb{Y}^2_{C_6}$ contains familiar subspaces; can't you achieve a similar result through a multipolar generalization of quaternions constructed using ordered pairs of complex hexapolars?

Yes, one may construct a multipolar generalization of quaternions with ordered pairs of complex hexapolars as per the Cayley–Dickson construction in which multiplication is defined such that for $a, b, c, d \in \mathbb{Y}_{C_6}$, $(a, b)(c, d) = (ac + -(d^*b), da + bc^*)$. Whereas the imaginary basis vectors of the quaternions i, j, k each square to -1 such that $ijk = -1$, the imaginary-like basis vectors of hexapolar quaternions

TABLE 7. Hexapolar Quaternion Multiplication Table

	1	s	j	\$
1	1	s	j	\$
s	s	S1	\$	Sj
j	j	$-$\$	-1	s
\$	\$	Pj	$-s$	S1

$s, j, \$$ have distinguishing properties as per the multiplication table depicted such that $sj\$ = \text{S1}$. Conjugation of the hexapolar quaternion (a, b) may be defined such that $(a, b)^* = (a^*, -b)$ so that for $(a, b)(c, d)^* = (f, g)$, a generalized inner product operation $\langle (a, b), (c, d) \rangle = (f, 0)$.

10.1. **Hexapolar quaternion multiplication seems off-putting due to its computational complexity. Anything noteworthy in terms of closure, algebraic closure, nilpotent elements, or computational options?** Yes, the hexapolar quaternions are closed under multiplication, though they are not algebraically closed in a strict sense because they do not form a field. The hexapolar quaternions do contain nilpotent elements; to apprehend how this happens, attend to the role of conjugation in the expression of multiplication above with the reminder that numbers $\in \mathbf{L}_1$ or $\in \mathbf{L}_2$ annihilate when multiplied with their conjugates (e.g., $(0, 1 + s)$). Alternately, observe the cancellation of component terms in matrix representation of the hexapolar quaternions.

10.2. **How can I represent the hexapolar quaternions using generalized Pauli matrices?** One may generalize the Pauli matrices using complex hexapolar components and hexapolar coefficients as per the matrix representation below. For $a, b, c, d \in M_{C_6}$, one may represent a hexapolar quaternion as

$$a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & s \\ s & 0 \end{bmatrix} + c \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} s & 0 \\ 0 & -s \end{bmatrix}.$$

For example, the nilpotent hexapolar quaternion (of index/degree two) expressed as $(0, 1 + s)$ above may also be expressed as $\begin{bmatrix} s & -1 \\ 1 & -s \end{bmatrix}$.

10.3. **Four basis vectors and six signs! What kind of multiplicative group structure does that instantiate or imply?** Quaternion units and hexapolar quaternion units are nonabelian groups under multiplication; one may generalize the quaternion group of order eight from Lipschitz quaternion units (two signs, four basis vectors) to the group of 24 hexapolar quaternion units (six signs, four basis vectors) such that the latter group is isomorphic to the binary tetrahedral group 2T and the special linear group SL(2,3)[10].

10.4. **Something smells a little fishy with the idea of hexapolar versors; what's the catch?** Hexapolar quaternion multiplication expressed as exponentiation functions of θ indicates that, like $i^{2\theta/\pi}$, $j^{2\theta/\pi}$ may be periodic in increments of 2π and that $s^{2\theta/\pi}$ & $\$^{2\theta/\pi}$ may each be periodic in increments of 6π.

11. I know that a scalene triangle (a figure having no two edges of equal length) may be embedded in the complex plane and so may be regarded as a chiral figure. However, that same figure may not be regarded as chiral if embedded in the quaternion extension of that plane (i.e., in a four-dimensional dipolar space); if a figure chiral in the complex plane is also embedded in a hexapolar extension of that plane, may that figure necessarily remain chiral?

A figure chiral in the complex plane also embedded in a multipolar extension of that plane does not necessarily remain chiral.

Assuming a space in which axes have only two poles (i.e., a dipolar space), the chirality of an object embedded therein depends on a relation between its minimum embedding dimension and the dimension of the space in which it is embedded[3][8]. Relaxing or suspending the constraint on the maximum number of axial poles permits the mapping of mirror-symmetric objects through rigid transformation without requiring increase in the dimension of the embedding space. For example, a chiral figure in the complex plane such as a scalene triangle may be brought to coincide with its mirror image by embedding both in a multipolar plane such that their hyperplane of reflection coincides with a coordinate axis of the space and they are translated along an orthogonal axis in a direction mutually inverse to those that define the orthants containing the pair of figures.

For example, figures in orthants (+,+) and (S,+) that reflect each other over the imaginary-like axis may be translated by, say, (-4,0) to orthant ($-$,+) and thus be brought to coincide. More specifically, the set of points $\{1+s,\ 2+s,\text{ and } 3+3s\}$ may be mirrored by the set of points $\{\mathrm{S}1+s,\ \mathrm{S}2+s,\text{ and } \mathrm{S}3+3s\}$ such that adding -4 to each point in both sets of points results in one set of points: $\{-3+s,\ -2+s,$ and $-1+3s\}$; the mirror images so translated coincide in the plane (i.e., without rotation given an additional dimension or basis vector). (It may be worth noting that such a translation seems to have the effect of reversing the orientation of the point-sets relative to their former axis of reflection, in this case the imaginary-like axis.)

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