COMPLEX HEXAPOLAR ROOTS AND 2X2 MATRIX REPRESENTATION: AN OVERVIEW *DRAFT*

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ABSTRACT. Draft, TBD.

CONTENTS

1. INTRODUCTION AND MOTIVATION

The complex hexapolar numbers ($\mathbb{Y}_{C_6}$) are an instance of a complex multipolar number system ($\mathbb{Y}_{C_p}$) that extends the complex numbers ($\mathbb{C}$) by including the minimum quantity of signs > 2 required to embed the real line as per a generalization of addition that does not necessarily associate (consolidation) such that additive inverse pairs are not necessarily unique. (A formal treatment of their construction and an expository overview each appear elsewhere[2]; the purpose of this document is to discuss an extension of that number system.) All nonzero numbers in $\mathbb{Y}_{C_6}$ are contained by at least one of three intersecting subsets ($\mathbf{L}_0$, $\mathbf{L}_1$, $\mathbf{L}_2$) that each form a group under multiplication (i.e., elements of those subsets associate and have inverses) such that $\mathbb{C} \in \mathbf{L}_0$ and $\mathbb{C} \neq \mathbf{L}_0$. Apart from the four quadrants of $\mathbb{C}$, $\mathbf{L}_0$ contains eight so-called rootless ray orthants; their name refers to the fact that each such orthant contains a ray from the origin of unbounded magnitude such that its constituent points may be expressed as components of equal magnitude and each such number has no square root $\in \mathbb{Y}_{C_6}$. Eight elements of unit magnitude when squared (for scaling to arbitrary distance from the origin along a given rootless ray in the complex hexapolar plane) may be expressed as 2x2 matrices of complex hexapolar numbers as per a generalized matrix multiplication that also admits a non-overlapping set of multiplicative inverses of those eight elements; each of those two eight-element sets contain four pairs of inverses under a generalized matrix addition (component-wise consolidation). A given root matrix produces the same

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number as its transpose when squared. Under a generalized Frobenius inner product operation, such elements are orthogonal to themselves and their inverses under addition and multiplication.

TABLE 1. Rootless Rays $\in \mathbb{Y}_{C_6}$ By Component Signs a, b & Their Roots

	T,−	H,−	H,+	T,+
$\Upsilon_{a,b}$	$\begin{bmatrix} 0 & e^{i(\pi/4)} \\ s^4 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & e^{i(3\pi/4)} \\ s^4 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & e^{i(5\pi/4)} \\ s^4 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & e^{i(7\pi/4)} \\ s^4 & 0 \end{bmatrix}$
$\Upsilon^{\intercal}_{a,b}$	$\begin{bmatrix} 0 & s^4 \\ e^{i(\pi/4)} & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & s^4 \\ e^{i(3\pi/4)} & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & s^4 \\ e^{i(5\pi/4)} & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & s^4 \\ e^{i(7\pi/4)} & 0 \end{bmatrix}$
	P,H	S,H	S,T	P,T
$\Upsilon_{a,b}$	$\begin{bmatrix} 0 & e^{i(\pi/4)} \\ s^8 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & e^{i(3\pi/4)} \\ s^8 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & e^{i(5\pi/4)} \\ s^8 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & e^{i(7\pi/4)} \\ s^8 & 0 \end{bmatrix}$
$\Upsilon^{\intercal}_{a,b}$	$\begin{bmatrix} 0 & s^8 \\ e^{i(\pi/4)} & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & s^8 \\ e^{i(3\pi/4)} & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & s^8 \\ e^{i(5\pi/4)} & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & s^8 \\ e^{i(7\pi/4)} & 0 \end{bmatrix}$

TABLE 2. Upsilon Root Matrices as Unique Pairs of Inverses

	$\Upsilon_{a,b} \vert \Upsilon^2_{a,b} \approx a.7 + b.7s$	$+ = 0$	$\times = 1$
1	$\Upsilon_{\mathrm{T},-} = \begin{bmatrix} 0 & e^{i(\pi/4)} \\ s^4 & 0 \end{bmatrix}$	$\Upsilon_{\mathrm{S,T}}$	$\Upsilon^{\intercal}_{\mathrm{P,T}}$
2	$\Upsilon_{\mathrm{S,T}} = \begin{bmatrix} 0 & e^{i(5\pi/4)} \\ s^8 & 0 \end{bmatrix}$	$\Upsilon_{\mathrm{T},-}$	$\Upsilon^{\intercal}_{\mathrm{H},-}$
3	$\Upsilon^{\intercal}_{\mathrm{P,T}} = \begin{bmatrix} 0 & s^8 \\ e^{i(7\pi/4)} & 0 \end{bmatrix}$	$\Upsilon^{\intercal}_{\mathrm{H},-}$	$\Upsilon_{\mathrm{T},-}$
4	$\Upsilon^{\intercal}_{\mathrm{H},-} = \begin{bmatrix} 0 & s^4 \\ e^{i(3\pi/4)} & 0 \end{bmatrix}$	$\Upsilon^{\intercal}_{\mathrm{P,T}}$	$\Upsilon_{\mathrm{S,T}}$
5	$\Upsilon_{\mathrm{H},-} = \begin{bmatrix} 0 & e^{i(3\pi/4)} \\ s^4 & 0 \end{bmatrix}$	$\Upsilon_{\mathrm{P,T}}$	$\Upsilon^{\intercal}_{\mathrm{S,T}}$
6	$\Upsilon_{\mathrm{P,T}} = \begin{bmatrix} 0 & e^{i(7\pi/4)} \\ s^8 & 0 \end{bmatrix}$	$\Upsilon_{\mathrm{H},-}$	$\Upsilon^{\intercal}_{\mathrm{T},-}$
7	$\Upsilon^{\intercal}_{\mathrm{S,T}} = \begin{bmatrix} 0 & s^8 \\ e^{i(5\pi/4)} & 0 \end{bmatrix}$	$\Upsilon^{\intercal}_{\mathrm{T},-}$	$\Upsilon_{\mathrm{H},-}$
8	$\Upsilon^{\intercal}_{\mathrm{T},-} = \begin{bmatrix} 0 & s^4 \\ e^{i(\pi/4)} & 0 \end{bmatrix}$	$\Upsilon^{\intercal}_{\mathrm{S,T}}$	$\Upsilon_{\mathrm{P,T}}$
9	$\Upsilon_{\mathrm{T},+} = \begin{bmatrix} 0 & e^{i(7\pi/4)} \\ s^4 & 0 \end{bmatrix}$	$\Upsilon_{\mathrm{S,H}}$	$\Upsilon^{\intercal}_{\mathrm{P,H}}$
10	$\Upsilon_{\mathrm{S,H}} = \begin{bmatrix} 0 & e^{i(3\pi/4)} \\ s^8 & 0 \end{bmatrix}$	$\Upsilon_{\mathrm{T},+}$	$\Upsilon^{\intercal}_{\mathrm{H},+}$
11	$\Upsilon^{\intercal}_{\mathrm{P,H}} = \begin{bmatrix} 0 & s^8 \\ e^{i(\pi/4)} & 0 \end{bmatrix}$	$\Upsilon^{\intercal}_{\mathrm{H},+}$	$\Upsilon_{\mathrm{T},+}$
12	$\Upsilon^{\intercal}_{\mathrm{H},+} = \begin{bmatrix} 0 & s^4 \\ e^{i(5\pi/4)} & 0 \end{bmatrix}$	$\Upsilon^{\intercal}_{\mathrm{P,H}}$	$\Upsilon_{\mathrm{S,H}}$
13	$\Upsilon^{\intercal}_{\mathrm{T},+} = \begin{bmatrix} 0 & s^4 \\ e^{i(7\pi/4)} & 0 \end{bmatrix}$	$\Upsilon^{\intercal}_{\mathrm{S,H}}$	$\Upsilon_{\mathrm{P,H}}$
14	$\Upsilon^{\intercal}_{\mathrm{S,H}} = \begin{bmatrix} 0 & s^8 \\ e^{i(3\pi/4)} & 0 \end{bmatrix}$	$\Upsilon^{\intercal}_{\mathrm{T},+}$	$\Upsilon_{\mathrm{H},+}$
15	$\Upsilon_{\mathrm{P,H}} = \begin{bmatrix} 0 & e^{i(\pi/4)} \\ s^8 & 0 \end{bmatrix}$	$\Upsilon_{\mathrm{H},+}$	$\Upsilon^{\intercal}_{\mathrm{T},+}$
16	$\Upsilon_{\mathrm{H},+} = \begin{bmatrix} 0 & e^{i(5\pi/4)} \\ s^4 & 0 \end{bmatrix}$	$\Upsilon_{\mathrm{P,H}}$	$\Upsilon^{\intercal}_{\mathrm{S,H}}$

2. How the complex hexapolars and their upsilon roots relate to complex ($\mathbb{C}$) and real ($\mathbb{R}$) numbers

For $i \in \mathbb{C}$ and $s \in \mathbb{Y}_{C_6}$, $s^3 = i$ such that $s \notin \mathbb{C}$; $s \neq -i = s^9$. $s^{12} = 1, s^6 = -1$. In general, for s^n such that $n \in \mathbb{N}$ and $n \bmod 2 = 1$, s^n is an additive inverse of an imaginary or imaginary-like element whereas for s^m such that $m \in \mathbb{N}$ and $m \bmod 2 = 0$, s^m is an additive inverse of a real or real-like element. Certain upsilon matrices (and their transposes) may be fourth roots of $s, s^5, s^7, s^{11} \in \mathbb{Y}_{C_6}$ as indicated in the table below (and thus higher-power roots of unity and its inverses under consolidation/generalized addition).

Table 3. Conversion between elements of $\mathbb{R}, \mathbb{C}, \mathbb{Y}_{C_6}$, and Υ_{C_6}

Number System												
$\mathbb{R}$	-	-	-	-	-	-1	-	-	-	-	-	1
$\mathbb{C}$	-	-	i	-	-	-1	-	-	$-i$	-	-	1
$\mathbb{C}$ (as i^n)	-	-	i	-	-	i^2	-	-	i^3	-	-	i^4
$\mathbb{Y}_{C_6}$	s	S1	Ss	T1	Ts	-1	$-s$	P1	Ps	H1	Hs	1
$\mathbb{Y}_{C_6}$ (as s^n)	s^1	s^2	s^3	s^4	s^5	s^6	s^7	s^8	s^9	s^{10}	s^{11}	s^{12}
Υ_{C_6} (as $\Upsilon^n_{a,b}$)	$\Upsilon^4_{\mathrm{P,T}}$				$\Upsilon^4_{\mathrm{H},-}$		$\Upsilon^4_{\mathrm{P,H}}$				$\Upsilon^4_{\mathrm{T},-}$	
Υ_{C_6} (as $\Upsilon^n_{a,b}$)	$\Upsilon^4_{\mathrm{S,H}}$				$\Upsilon^4_{\mathrm{T},+}$		$\Upsilon^4_{\mathrm{S,T}}$				$\Upsilon^4_{\mathrm{H},+}$	

Table 4. Sign multiplication table of $\mathbb{Y}_{C_6}$

	+1	**S1**	**T1**	**-1**	**P1**	**H1**
+1	+1	S1	T1	-1	P1	H1
S1	S1	T1	-1	P1	H1	+1
T1	T1	-1	P1	H1	+1	S1
-1	-1	P1	H1	+1	S1	T1
P1	P1	H1	+1	S1	T1	-1
H1	H1	+1	S1	T1	-1	P1

3. Do the roots that extend the complex hexapolars have roots?

Yes, a 2x2 matrix of complex hexapolar components may express an object that equals an upsilon matrix upon multiplication with itself.

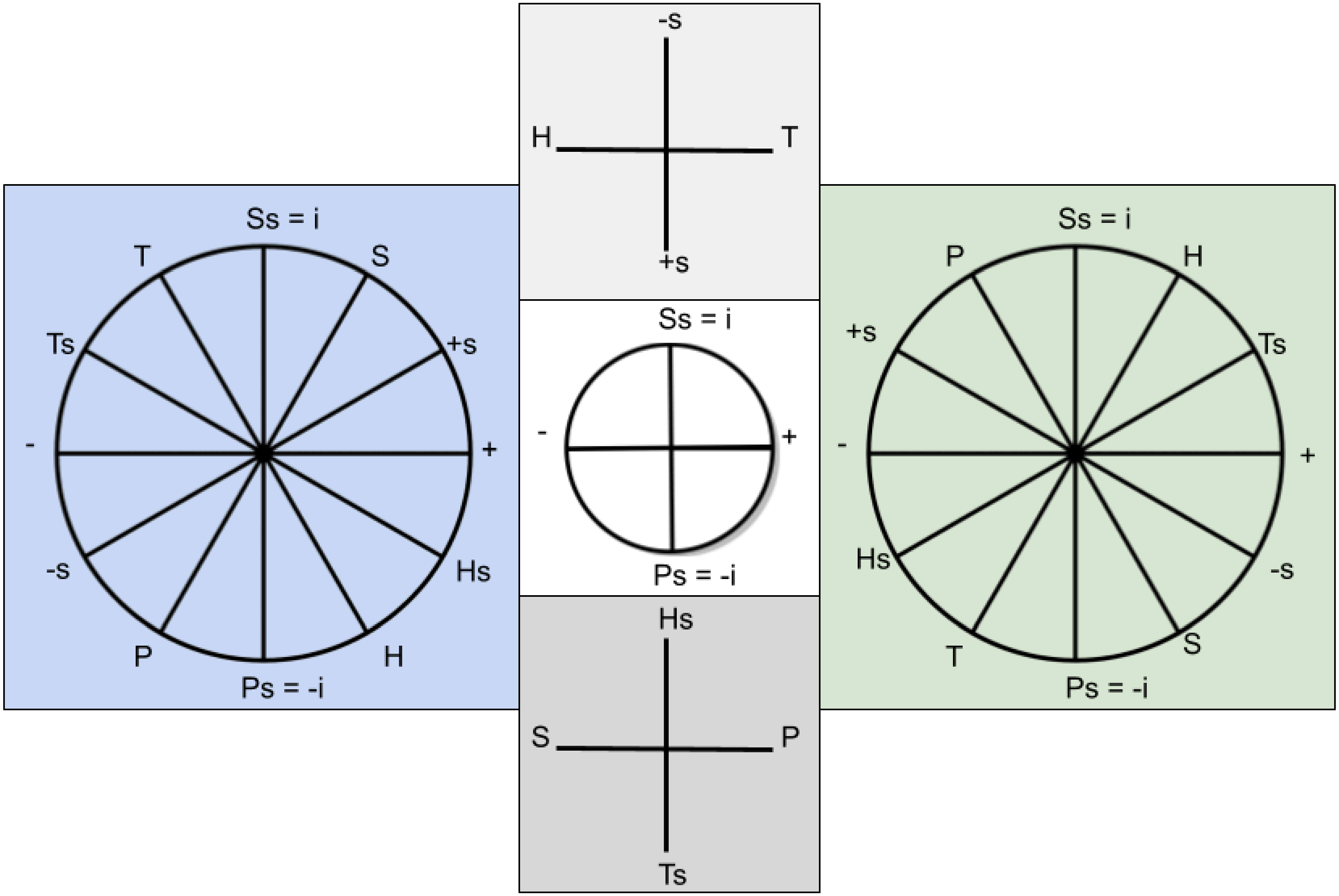
For example, $\Upsilon_{\mathrm{T},-} = \begin{bmatrix} 0 & e^{i(\pi/4)} \\ s^4 & 0 \end{bmatrix}$.

$$\begin{bmatrix} (s^8)e^{i(\pi/8)}/\sqrt{2(s^8)e^{i(\pi/8)}} & e^{i(\pi/4)}/\sqrt{2(s^8)e^{i(\pi/8)}} \\ s^4/\sqrt{2(s^8)e^{i(\pi/8)}} & (s^8)e^{i(\pi/8)}/\sqrt{2(s^8)e^{i(\pi/8)}} \end{bmatrix}^2 = \begin{bmatrix} 0 & e^{i(\pi/4)} \\ s^4 & 0 \end{bmatrix}.$$

Examine the expressions above in relation to $\Upsilon_{\mathrm{P,H}} = \begin{bmatrix} 0 & e^{i(\pi/4)} \\ s^8 & 0 \end{bmatrix}$ such that

$$\begin{bmatrix} (s^4)e^{i(\pi/8)}/\sqrt{2(s^4)e^{i(\pi/8)}} & e^{i(\pi/4)}/\sqrt{2(s^4)e^{i(\pi/8)}} \\ s^8/\sqrt{2(s^4)e^{i(\pi/8)}} & (s^4)e^{i(\pi/8)}/\sqrt{2(s^4)e^{i(\pi/8)}} \end{bmatrix}^2 = \begin{bmatrix} 0 & e^{i(\pi/4)} \\ s^8 & 0 \end{bmatrix}.$$

Visualization of the complex hexapolar plane
by Ben Blohowiak



4. Tables regarding $\mathbb{Y}_{C_6}$, its quaternion extension, and $\mathbb{Y}^2_{C_6}$

Tables that might make for helpful reference material. Tables that might make for helpful reference material. Tables that might make for helpful reference material. Tables that might make for helpful reference material.Tables that might make for helpful reference material. Tables that might make for helpful reference material.Tables that might ma

Table 5. Equivalence Classes of 4-Orthant Sets (orthants indicated by signed unit pairs)

	$f(\theta)$	(a,b)	$i(a,b)$	$-1(a,b)$	$(-i)(a,b)$
$\in \mathbb{C}$	$(s^3)^{2\theta/\pi}$	(+1,P1)	(+1,S1)	(−1,S1)	(−1,P1)
$\notin \mathbb{C}$	-	(T1,+1)	(T1,−1)	(H1,−1)	(H1,+1)
$\notin \mathbb{C}$	-	(P1,T1)	(P1,H1)	(S1,H1)	(S1,T1)
$\notin \mathbb{C}$	$s^{2\theta/\pi}$	(+1,+1)	(T1,S1)	(−1,−1)	(H1,P1)
$\notin \mathbb{C}$	$s^{2\theta/\pi}$	(T1,T1)	(P1,−1)	(H1,H1)	(S1,+1)
$\notin \mathbb{C}$	$s^{2\theta/\pi}$	(P1,P1)	(+1,H1)	(S1,S1)	(−1,T1)
$\notin \mathbb{C}$	$(s^5)^{2\theta/\pi}$	(+1,T1)	(P1,S1)	(−1,H1)	(S1,P1)
$\notin \mathbb{C}$	$(s^5)^{2\theta/\pi}$	(P1,+1)	(T1,H1)	(S1,−1)	(H1,T1)
$\notin \mathbb{C}$	$(s^5)^{2\theta/\pi}$	(T1,P1)	(+1,−1)	(H1,S1)	(−1,+1)

Table 6. $f(\theta)$: Exponentiation of s (orthants indicated by signed unit pairs)

	θ **Range**	$s^{2\theta/\pi}$	$(s^5)^{2\theta/\pi}$	$i^{2\theta/\pi}$	$(-i)^{2\theta/\pi}$
1	$(0,\pi/2)$	(+1,+1)	(+1,T1)	(+1,S1)	(+1,P1)
2	$(\pi/2,\pi)$	(S1,+1)	(H1,T1)	(−1,S1)	(−1,P1)
3	$(\pi,3\pi/2)$	(S1,S1)	(H1,S1)	(−1,P1)	(−1,S1)
4	$(3\pi/2,2\pi)$	(T1,S1)	(P1,S1)	(+1,P1)	(+1,S1)
5	$(2\pi,5\pi/2)$	(T1,T1)	(P1,+1)	(+1,S1)	(+1,P1)
6	$(5\pi/2,3\pi)$	(−1,T1)	(−1,+1)	(−1,S1)	(−1,P1)
7	$(3\pi,7\pi/2)$	(−1,−1)	(−1,H1)	(−1,P1)	(−1,S1)
8	$(7\pi/2,4\pi)$	(P1,−1)	(T1,H1)	(+1,P1)	(+1,S1)
9	$(4\pi,9\pi/2)$	(P1,P1)	(T1,P1)	(+1,S1)	(+1,P1)
10	$(9\pi/2,5\pi)$	(H1,P1)	(S1,P1)	(−1,S1)	(−1,P1)
11	$(5\pi,11\pi/2)$	(H1,H1)	(S1,−1)	(−1,P1)	(−1,S1)
12	$(11\pi/2,6\pi)$	(+1,H1)	(+1,−1)	(+1,P1)	(+1,S1)

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Table 7. Example products of complex hexapolars with their conjugates

	Polar form $f(r,\theta)$	$a = r(s^x)$	$b = r(s^y)$	$(a,b)(a,b)^*$	$\|\|$ if $b/s^y \neq r$
$\in \mathbb{C}$	$((2r^2)^{1/2})(s^3)^{2\theta/\pi}$	rs^0 or rs^6	rs^2 or rs^8	$(\|(a,b)\|^2)(\mathrm{sgn}(a)^2)$	$r^2 + r'^2$
$\notin \mathbb{C}$	-	rs^2 or rs^8	rs^{10} or rs^4	$(\|(a,b)\|^2)(\mathrm{sgn}(a)^2)$	$r^2 + r'^2$
$\notin \mathbb{C}$	-	rs^4 or rs^{10}	rs^0 or rs^6	$(\|(a,b)\|^2)(\mathrm{sgn}(a)^2)$	$r^2 + r'^2$
$\notin \mathbb{C}$	$((2r^2)^{1/2})s^{2\theta/\pi}$	rs^0	rs^0 or rs^{10}	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})s^{2\theta/\pi}$	rs^2	rs^0 or rs^2	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})s^{2\theta/\pi}$	rs^4	rs^2 or rs^4	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})s^{2\theta/\pi}$	rs^6	rs^4 or rs^6	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})s^{2\theta/\pi}$	rs^8	rs^6 or rs^8	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})s^{2\theta/\pi}$	rs^{10}	rs^8 or rs^{10}	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})(s^5)^{2\theta/\pi}$	rs^0	rs^4 or rs^6	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})(s^5)^{2\theta/\pi}$	rs^2	rs^6 or rs^8	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})(s^5)^{2\theta/\pi}$	rs^4	rs^8 or rs^{10}	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})(s^5)^{2\theta/\pi}$	rs^6	rs^{10} or rs^0	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})(s^5)^{2\theta/\pi}$	rs^8	rs^0 or rs^2	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})(s^5)^{2\theta/\pi}$	rs^{10}	rs^2 or rs^4	0	$\|r^2 - r'^2\|$

Table 8. $\mathbb{Y}^2_{C_6}$ Comprised by Planes A & B: Associations w/Subspaces

	$\mathbf{L}_0$**(A)**	$\mathbf{L}_1$ **or** $\mathbf{L}_2$ **(A)**
$\mathbf{L}_0$ **(B)**	$\mathbb{R}^4$	$\mathbb{R}^{1,3}$
$\mathbf{L}_1$ **or** $\mathbf{L}_2$ **(B)**	$\mathbb{R}^{1,3}$	$\mathbb{R}^{2,2}$

Table 9. Hexapolar Quaternion Multiplication Table

	1	s	j	\$
1	1	s	j	\$
s	s	S1	\$	Sj
j	j	$-$\$	-1	s
\$	\$	Pj	$-s$	S1

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