COMPLEX HEXAPOLAR NUMBERS AND THE COMPLEX HEXAPOLAR PLANE: AN OVERVIEW *DRAFT*

BEN B. BLOHOWIAK

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Contents

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1. Introduction and Motivation

The complex hexapolar numbers ($\mathbb{Y}_{C_6}$) are an instance of a complex multipolar number system ($\mathbb{Y}_{C_p}$) that extends the complex numbers ($\mathbb{C}$). (A more formal treatment of their construction appears elsewhere[1]; the purpose of this document is generally expository and for quick reference.) The complex hexapolars are the minimum nontrivial multipolar embedding of the linear continuum of the reals ($\mathbb{R}$) and its extension by an imaginary fourth root of unity (i); $\mathbb{R} \in \mathbb{C} \in \mathbb{Y}_{C_p} \iff (p/2)\text{mod}2 = 1$ and $p \geq 6$. Whereas quaternions ($\mathbb{H}$) extend and embed $\mathbb{C}$ such that multiplication is not necessarily commutative for $a \in \mathbb{H}$ such that $a \notin \mathbb{C}$, multipolar addition is not necessarily associative for expressions containing more than two unlike signs ($\mathbb{Y}_{C_6}$ contains a total of six signs, four of which $\notin \mathbb{C}$). For $\mathbb{Y}_{C_p}$, which may be represented as an ordered pair of multipolar numbers, corresponding components of unlike sign and equal magnitude annihilate under a generalized addition operation which distributes over multiplication.

2. How the complex hexapolars relate to complex ($\mathbb{C}$) and real ($\mathbb{R}$) numbers

For $i \in \mathbb{C}$ and $s \in \mathbb{Y}_{C_6}$, $s^3 = i$ such that $s \notin \mathbb{C}$; $s \neq -i = s^9$. $s^{12} = 1, s^6 = -1$. In general, for s^n such that $n \in \mathbb{N}$ and $n\text{mod}2 = 1$, s^n is an additive inverse of an imaginary or imaginary-like element whereas for s^m such that $m \in \mathbb{N}$ and $m\text{mod}2 = 0$, s^m is an additive inverse of a real or real-like element.

Table 1. Conversion between elements of $\mathbb{R}, \mathbb{C}$, and $\mathbb{Y}_{C_6}$

Number System													
$\mathbb{R}$	0	-	-	-	-	-	-1	-	-	-	-	-	1
$\mathbb{C}$	0	-	-	i	-	-	-1	-	-	$-i$	-	-	1
$\mathbb{C}$ (as i^n)	0	-	-	i	-	-	i^2	-	-	i^3	-	-	i^4
$\mathbb{Y}_{C_6}$	0	s	S1	Ss	T1	Ts	-1	$-s$	P1	Ps	H1	Hs	1
$\mathbb{Y}_{C_6}$ (as s^n)	0	s^1	s^2	s^3	s^4	s^5	s^6	s^7	s^8	s^9	s^{10}	s^{11}	s^{12}

3. What is the complex hexapolar plane? What does that even mean?

As the complex plane is a geometric interpretation of the complex number system, the complex hexapolar plane is a geometric interpretation of the complex hexapolars. The complex hexapolar plane embeds the complex plane; one may also say that it is the complex plane extended by other elements. Whereas quaternions extend the complex plane with two imaginary basis vectors that introduce an isometry with a four-dimensional vector space over the real numbers, complex hexapolars contain the same quantity of basis vectors as the complex numbers (two:

TABLE 2. Sign multiplication table of $\mathbb{Y}_{C_6}$

	+1	**S1**	**T1**	**-1**	**P1**	**H1**
+1	+1	S1	T1	-1	P1	H1
S1	S1	T1	-1	P1	H1	+1
T1	T1	-1	P1	H1	+1	S1
-1	-1	P1	H1	+1	S1	T1
P1	P1	H1	+1	S1	T1	-1
H1	H1	+1	S1	T1	-1	P1

corresponding components of unlike sign are linearly dependent), and so one speaks in terms of a two-dimensional plane. The imaginary axis is extended by four classes of elements that are inverses to each other, i, and $-i$ under a generalized addition; this implies a real axis extended by four classes of elements that are inverses to each other, 1, and -1 under a generalized addition.

Whereas the complex plane contains two axes each containing two mutually inverse directions of extension π radians apart, each axis of the complex hexapolar plane contains a total of six mutually inverse directions of extension that may be computed π radians apart using a generalized inner product operation. The complex plane, with its two signs and two axes (2^2), contains four quadrants (or, more generally, four orthants) and the complex hexapolar plane, with its six signs and two axes (6^2), contains 36 orthants, four of which belong to the complex plane.

3.1. **36 orthants instead of the usual four quadrants and the total of 36 includes the usual four quadrants? How does that work?** A heuristic for building intuitions regarding the structure of the complex hexapolar plane might be to think of it as the complex plane extended by two sets of metaphorically twinned regions that all share an origin such that $\mathbb{Y}_{C_6}$ could be said to consist of:

- The complex plane (the four quadrants/orthants $\in \mathbb{C}$)
- Eight so-called *ghost orthants* (which may be characterized as a pair of isometric four-quadrant analogues of the complex plane) whose points may not be expressed in polar form (e.g., $\neq (s^x)^{2\theta/\pi}$ and $\notin \mathbb{C}$)
- 24 so-called *long-cycle orthants* (which may be characterized as a pair of isometric 12-orthant regions) whose points are $\notin \mathbb{C}$ and may be expressed in polar form

4. How could someone enhance their intuitions regarding the structure of (or visualize) the complex hexapolar plane?

Many approaches are possible; some involve representing a point on an axis, such as the origin, in more than one location on a map, chart, or depiction.

One way to represent the 36 orthants of the complex hexapolar plane is with a set of nine squares such that each square contains four quadrants (see table of equivalence classes of four-orthant sets). Despite the clumsy verbalization below, the combinatorics are straightforward: chunk the 36 orthants into nine four-quadrant sets ("squares") so that they resemble nine versions of $\mathbb{R}^2$. One such four-quadrant square corresponds to the complex plane. The nine squares may be grouped into three sets as per their shared signs polarities of the imaginary axis such that a given

Visualization of the complex hexapolar plane
by Ben Blohowiak

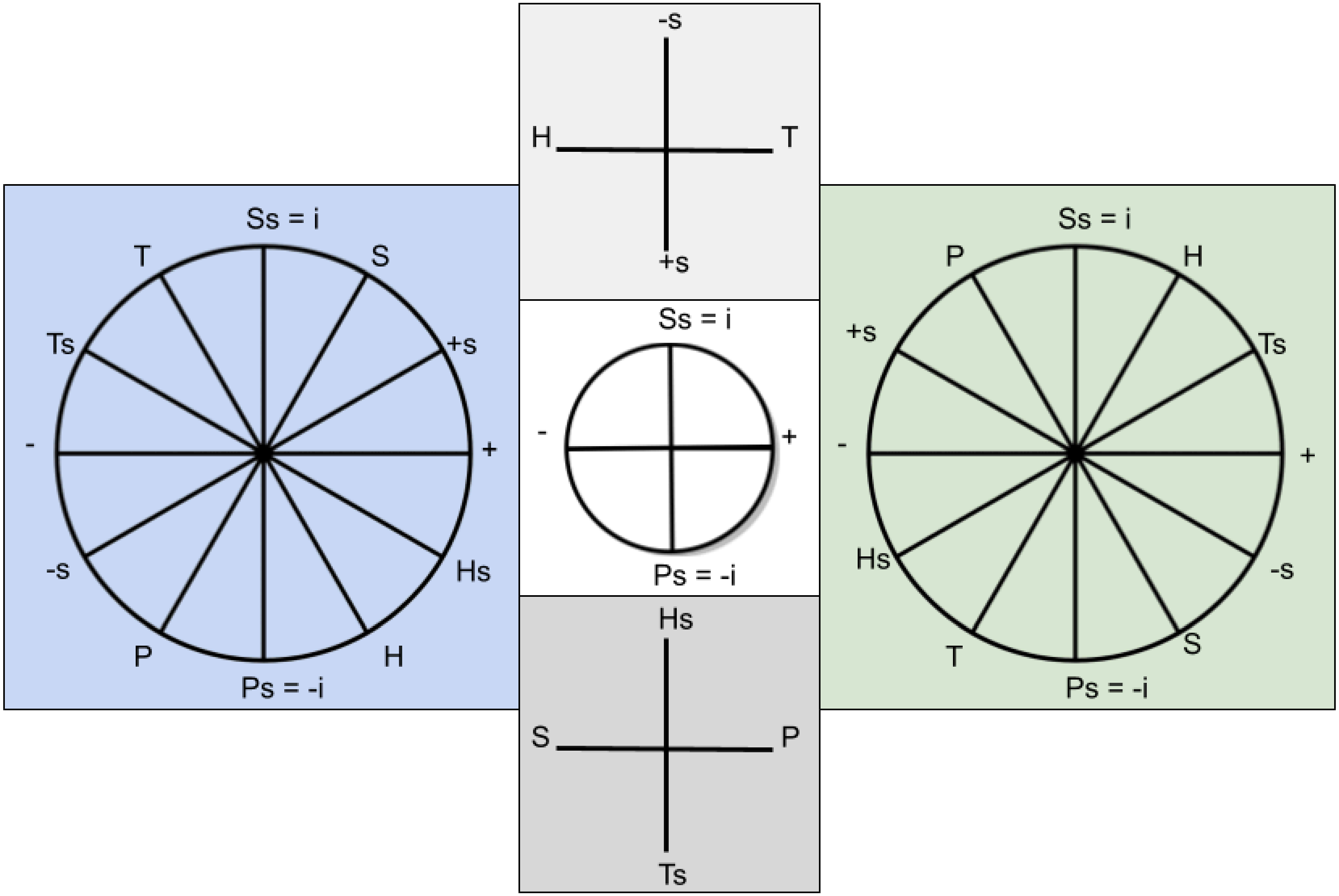

TABLE 3. Equivalence Classes of 4-Orthant Sets (orthants indicated by signed unit pairs)

	$f(\theta)$	(a,b)	$i(a,b)$	$-1(a,b)$	$(-i)(a,b)$
$\in \mathbb{C}$	$(s^3)^{2\theta/\pi}$	(+1,P1)	(+1,S1)	(−1,S1)	(−1,P1)
$\notin \mathbb{C}$	-	(T1,+1)	(T1,−1)	(H1,−1)	(H1,+1)
$\notin \mathbb{C}$	-	(P1,T1)	(P1,H1)	(S1,H1)	(S1,T1)
$\notin \mathbb{C}$	$s^{2\theta/\pi}$	(+1, +1)	(T1,S1)	(−1,−1)	(H1,P1)
$\notin \mathbb{C}$	$s^{2\theta/\pi}$	(T1,T1)	(P1,−1)	(H1,H1)	(S1,+1)
$\notin \mathbb{C}$	$s^{2\theta/\pi}$	(P1,P1)	(+1,H1)	(S1,S1)	(−1,T1)
$\notin \mathbb{C}$	$(s^5)^{2\theta/\pi}$	(+1,T1)	(P1,S1)	(−1,H1)	(S1,P1)
$\notin \mathbb{C}$	$(s^5)^{2\theta/\pi}$	(P1,+1)	(T1,H1)	(S1,−1)	(H1,T1)
$\notin \mathbb{C}$	$(s^5)^{2\theta/\pi}$	(T1,P1)	(+1, −1)	(H1,S1)	(−1,+1)

axial point is represented in triplicate (e.g., the sign polarity of the imaginary axis of the complex plane may be expressed in terms of s^3 (Ss) and s^9 (Ps) and, as well as the quadrants they define with the real axis sign polarity pair of +1 and -1, the inverse directions of extension of the imaginary axis identified by s^3 and s^9 also co-define a total of eight other quadrants in two other squares co-defined by regions of a real-like axis π radians away from +1 and -1 (e.g., S1, P1, etc.)).

Another way to represent the 36 orthants of the complex hexapolar plane is with a set of 5 squares such that there is a set of three squares containing four orthants each and a set of two squares containing twelve orthants each. The set of three squares contains one corresponding to the complex plane; the other two of the set of three squares correspond to regions of the complex hexapolar plane that may be arrived at from a non-origin point in the complex plane via scalar multiplication by s^4 or s^8 (the *ghost orthants* that have no polar expression). In total, those three squares contain 12 orthants. The remaining two squares each correspond to 12 orthants of the complex hexapolar plane and may be scaled proportionally (i.e., the area of one square of the set of two squares as equal in total area to the set of three squares). Representing 12 orthants in the four quadrants of each of the set of two squares may proceed according to the mappings $s^{2\theta/\pi} \mapsto i^{2\theta/3\pi}$ and $(s^5)^{2\theta/\pi} \mapsto i^{2\theta/3\pi}$, respectively. In such mappings, axial directions away from the origins of each of the squares in the set of two squares resemble the cutting lines of a 12-slice pie, with axes mapped such that they project as alternating real and imaginary "spokes" extending as if from the central hub of the origin (see attached visualization).

5. I know that each of the three vertices of a scalene triangle (a figure for which no two sides are of equal length) may be represented as points on the complex plane by mapping each vertex to a complex number such that it may be regarded as a chiral figure, however I also know that that same figure embedded in the complex plane may not be regarded as chiral if it is also embedded in the quaternion extension of that plane such that it is the same up to isomorphism as a four-dimensional space; if a figure chiral in the complex plane is also embedded in a multipolar extension of that plane, may that figure necessarily remain chiral?

A figure chiral in the complex plane also embedded in a multipolar extension of that plane might not necessarily remain chiral.

Assuming a space in which axes have only two poles, the chirality of an object embedded therein depends on a relation between its minimum embedding dimension and the dimension of the space in which it is embedded[2][7]; relaxing the constraint on the maximum number of axial poles permits the mapping of mirror-symmetric objects through rigid transformation without requiring increase in the dimension of the embedding space. For example, a chiral figure in the complex plane such as a scalene triangle may be brought to coincide with its mirror image by embedding both in a multipolar plane such that their hyperplane of reflection coincides with a coordinate axis of the space and they are translated along an orthogonal axis in a direction mutually inverse to those that define the orthants containing the pair of figures (e.g., figures reflected over the imaginary axis in orthants (+,+) and (S,+) may be translated by, say, (−9,0) to orthant (−,+) and thus be brought to coincide).

6. The unit circle of the complex plane is a useful and beautiful mathematical object. Anything noteworthy regarding the unit circle of the complex hexapolars?

The unit circle of the complex plane is embedded in the complex hexapolars ($i^{2\theta/\pi} = (s^3)^{2\theta/\pi}$; $(-i)^{2\theta/\pi} = (s^9)^{2\theta/\pi}$). What may be of interest is that if a unit circle of the complex plane is defined as the set A of elements $a \in \mathbb{C}$ such that $|a| = 1$ and the unit circle of the complex hexapolar plane is defined as the set B of elements $b \in \mathbb{Y}_{C_6}$ such that $|b| = 1$, then $A \in B$ but $A \neq B$.

Unlike points in the complex plane alone, not all nonzero points in the complex hexapolar plane may be expressed in polar form. If the unit circle of the complex hexapolar plane is defined as the set D of elements that are outputs of functions of θ that exponentiate an imaginary-like unit as suggested above, $D \cap B$ but $D \neq B$; the set B contains elements of eight orthants (i.e., *ghost orthants*) whose points $\notin D$. The outputs of such functions of θ trace three distinct paths through subsets of the plane, one of which traverses the four quadrants of the complex plane such that the other two paths pass through twelve non-overlapping orthants (i.e., a set of twelve *long-cycle orthants*) each. In total, cycles of imaginary exponentiation pass through 28 orthants, bypassing 8 remaining orthants (i.e., *ghost orthants*).

6.1. **In the Exponentiation of s table, why only s and s^5 raised to the $2\theta/\pi$, what about the other powers of s?** s^{11} and s^7 trace the same paths as s and

TABLE 4. $f(\theta)$: Exponentiation of s (orthants indicated by signed unit pairs)

	θ **Range**	$s^{2\theta/\pi}$	$(s^5)^{2\theta/\pi}$	$i^{2\theta/\pi}$	$(-i)^{2\theta/\pi}$
1	$(0, \pi/2)$	(+1,+1)	(+1,T1)	(+1,S1)	(+1,P1)
2	$(\pi/2, \pi)$	(S1,+1)	(H1,T1)	(−1,S1)	(−1,P1)
3	$(\pi, 3\pi/2)$	(S1,S1)	(H1,S1)	(−1,P1)	(−1,S1)
4	$(3\pi/2, 2\pi)$	(T1,S1)	(P1,S1)	(+1,P1)	(+1,S1)
5	$(2\pi, 5\pi/2)$	(T1,T1)	(P1,+1)	(+1,S1)	(+1,P1)
6	$(5\pi/2, 3\pi)$	(−1,T1)	(−1,+1)	(−1,S1)	(−1,P1)
7	$(3\pi, 7\pi/2)$	(−1,−1)	(−1,H1)	(−1,P1)	(−1,S1)
8	$(7\pi/2, 4\pi)$	(P1,−1)	(T1,H1)	(+1,P1)	(+1,S1)
9	$(4\pi, 9\pi/2)$	(P1,P1)	(T1,P1)	(+1,S1)	(+1,P1)
10	$(9\pi/2, 5\pi)$	(H1,P1)	(S1,P1)	(−1,S1)	(−1,P1)
11	$(5\pi, 11\pi/2)$	(H1,H1)	(S1,−1)	(−1,P1)	(−1,S1)
12	$(11\pi/2, 6\pi)$	(+1,H1)	(+1,−1)	(+1,P1)	(+1,S1)

s^5, respectively, but in an opposing direction–analogous to $i^{2\theta/\pi}$ versus $(-i)^{2\theta/\pi}$. Powers of s that are evenly divisible by two behave differently when raised to $2\theta/\pi$ than do odd powers (e.g., $s^{12} = 1, s^6 = -1$). $s^3 = i$ and $s^9 = -i$; $(s^3)^{2\theta/\pi}$ and $(s^9)^{2\theta/\pi}$ trace in opposing directions the unit circle of the complex plane embedded in the complex hexapolar plane.

7. Do the properties of complex numbers in relation to their conjugates generalize to the other complex hexapolars $\notin \mathbb{C}$?

Some but not all properties of complex numbers in relation to their conjugates generalize without condition; some complex hexapolars exhibit novel properties in relation to their conjugates not expressed by conjugacy relations $\in \mathbb{C}$ as complex hexapolar conjugation is not necessarily distributive over addition or multiplication. Adding a given complex hexapolar with its conjugate annihilates any imaginary-like part and doubles the magnitude of any real-like part. Similarly, the geometric characterization of a number's conjugate as a reflection across the real-like axis also holds. However, if expressing complex hexapolars in polar form with $\theta, x \in \mathbb{R}_{\geq 0}$, the conjugate of $(s^x)^{2\theta/\pi}$ is not necessarily $(s^x)^{-2\theta/\pi}$, though that method of conjugation holds if $x = 3$ or $x = 9$, as such numbers are in the complex plane ($\mathbb{C} \in \mathbb{Y}_{C_6}$).

The product of a complex hexapolar with its conjugate is a real-like number (i.e., not an imaginary-like number). That product may be positive, nonpositive, or zero. Nonzero conjugate factors that annihilate under multiplication may contain components of unbounded magnitude and may be characterized by a pattern between their components such that a complex hexapolar (a, b) that will annihilate under multiplication with its conjugate may be expressed such that r is a positive real number, x, y are even natural numbers such that $x\text{mod}2 = 0$,ymod2=0, $a = r(s^x), b = r(s^y)$, and $(x-4)\text{mod}12 \neq y$ or $(x+2)\text{mod}12 \neq y$. This implies that for a given pair of values x and $x+6$, there is a pair of values y and $y+6$ that will not result in annihilation if (a, b) is multiplied by its conjugate; rather, its magnitude will be squared (such that it equals $2r^2$). In such cases, the coefficient

r of s^y may be substituted for a different positive real number r' such that the product magnitude equals $r^2 + r'^2$; such a substitution if $(x-4)\text{mod}12 \neq y$ or $(x+2)\text{mod}12 \neq y$ would result in a product neither zero nor of squared magnitude but rather of magnitude $|r^2 - r'^2|$.

TABLE 5. Example products of complex hexapolars with their conjugates

	Polar form $f(r,\theta)$	$a = r(s^x)$	$b = r(s^y)$	$(a,b)(a,b)^*$	$\|\|$ if $b/s^y \neq r$
$\in \mathbb{C}$	$((2r^2)^{1/2})(s^3)^{2\theta/\pi}$	rs^0 or rs^6	rs^2 or rs^8	$(\|(a,b)\|^2)(\text{sgn}(a)^2)$	$r^2 + r'^2$
$\notin \mathbb{C}$	-	rs^2 or rs^8	rs^{10} or rs^4	$(\|(a,b)\|^2)(\text{sgn}(a)^2)$	$r^2 + r'^2$
$\notin \mathbb{C}$	-	rs^4 or rs^{10}	rs^0 or rs^6	$(\|(a,b)\|^2)(\text{sgn}(a)^2)$	$r^2 + r'^2$
$\notin \mathbb{C}$	$((2r^2)^{1/2})s^{2\theta/\pi}$	rs^0	rs^0 or rs^{10}	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})s^{2\theta/\pi}$	rs^2	rs^0 or rs^2	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})s^{2\theta/\pi}$	rs^4	rs^2 or rs^4	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})s^{2\theta/\pi}$	rs^6	rs^4 or rs^6	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})s^{2\theta/\pi}$	rs^8	rs^6 or rs^8	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})s^{2\theta/\pi}$	rs^{10}	rs^8 or rs^{10}	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})(s^5)^{2\theta/\pi}$	rs^0	rs^4 or rs^6	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})(s^5)^{2\theta/\pi}$	rs^2	rs^6 or rs^8	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})(s^5)^{2\theta/\pi}$	rs^4	rs^8 or rs^{10}	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})(s^5)^{2\theta/\pi}$	rs^6	rs^{10} or rs^0	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})(s^5)^{2\theta/\pi}$	rs^8	rs^0 or rs^2	0	$\|r^2 - r'^2\|$
$\notin \mathbb{C}$	$((2r^2)^{1/2})(s^5)^{2\theta/\pi}$	rs^{10}	rs^2 or rs^4	0	$\|r^2 - r'^2\|$

7.1. **You referenced magnitude or distance from the origin; how might someone compute the magnitude of a complex hexapolar number or its analogue in a generalized inner product space?** Previously–in another piece[1]–, I defined a generalization of an inner product operation for multipolar numbers that preserves the intuitive notion that, say, $1+4s$ and 1+S4s are equidistant from the origin, as one might expect from a Euclidean metric. However, that generalization of inner product may apply to complex multipolar number systems that do not embed the complex plane ($\mathbb{C}$) and those that do alike; as an inner product space over $\mathbb{C}$ is conventionally defined such that the inner product of two complex numbers may be expressed as the product of one multiplied by the conjugate of the other, that avenue of generalizing the inner product operation to complex hexapolars does not necessarily lead to preservation of the intuitive notion that $1 + 4s$ and 1+S4s are equidistant from the origin. Rather, a generalized inner product based in conjugacy relations defines subset regions of the complex hexapolar plane whose points' distances from the origin resemble those of a space under a Euclidean metric (such as $\mathbb{C}$ and the eight so-called ghost orthants) and regions whose points' distances from the origin resemble those of a space under a Minkowski metric or split-complex plane, including points that are outputs of the functions of θ described above such that points of a "unit circle" so defined may vary in distance from the origin like conjugate unit hyperbolae. Unlike in the case of the split-complex numbers[3], however, each nonzero complex hexapolar has a multiplicative inverse.

7.2. **So with one generalized inner product operation, different regions of the complex hexapolar plane may fall under a Euclidean or Minkowski/hyperbolic norm? How do those disparate regions play together when two complex hexapolar planes are orthogonally juxtaposed, as in the case of $\mathbb{Y}^2_{C_6}$?** Similar to how one may associate $\mathbb{C}$ with $\mathbb{R}^2$ and likewise regions of $\mathbb{Y}_{C_6}$ with $\mathbb{R}^2$ or $\mathbb{R}^{1,1}$, $\mathbb{Y}^2_{C_6}$ embeds analogues of such planar subspaces as well as spaces associated with $\mathbb{R}^4$, $\mathbb{R}^{1,3}$, and $\mathbb{R}^{2,2}$. The combinatorics of such associations is expressed in the table below.

TABLE 6. $\mathbb{Y}^2_{C_6}$ Comprised by Planes A & B: Associations w/Subspaces

	$\mathbb{C}$ or Ghost Orthants (A)	**Long-Cycle Orthants (A)**
$\mathbb{C}$ or Ghost Orthants (B)	$\mathbb{R}^4$	$\mathbb{R}^{1,3}$
Long-Cycle Orthants (B)	$\mathbb{R}^{1,3}$	$\mathbb{R}^{2,2}$

TABLE 7. Hexapolar Quaternion Multiplication Table

	1	s	j	\$
1	1	s	j	\$
s	s	S1	\$	Sj
j	j	$-$\$	-1	s
\$	\$	Pj	$-s$	S1

7.3. $\mathbb{Y}^2_{C_6}$ **contains familiar subspaces; can't you achieve a similar result through a multipolar generalization of quaternions constructed using ordered pairs of complex hexapolars?** Yes, one may construct a multipolar generalization of quaternions with ordered pairs of complex hexapolars as per the Cayley–Dickson construction in which multiplication is defined such that for $a, b, c, d \in \mathbb{Y}_{C_6}$, $(a, b)(c, d) = (ac + -(d^*b), da + bc^*)$. Conjugation of the hexapolar quaternion (a, b) may be defined such that $(a, b)^* = (a^*, -b)$ so that for $(a, b)(c, d)^* = (f, g)$, a generalized inner product operation $\langle (a, b), (c, d) \rangle = (f, 0)$. Whereas the imaginary basis vectors of the quaternions i, j, k each square to -1 such that $ijk = -1$, the imaginary-like basis vectors of hexapolar quaternions $s, j, \$$ have distinguishing properties as per the multiplication table depicted such that $sj\$ =$ S1.

7.3.1. *Four basis vectors and six signs! What kind of multiplicative group structure does that instantiate or imply?* Quaternion units and hexapolar quaternion units are nonabelian groups under multiplication; one may generalize the quaternion group of order eight from Lipschitz quaternion units (two signs, four basis vectors) to the group of 24 hexapolar quaternion units (six signs, four basis vectors) such that the latter group is isomorphic to the binary tetrahedral group 2T and the special linear group SL(2,3)[9].

7.3.2. *Something smells a little fishy with the idea of hexapolar versors; what's the catch?* Hexapolar quaternion multiplication expressed as exponentiation functions of θ indicates that, like $i^{2\theta/\pi}$, $j^{2\theta/\pi}$ may be periodic in increments of 2π and that $s^{2\theta/\pi}$ & $\$^{2\theta/\pi}$ may each be periodic in increments of 6π.

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Charlottesville, VA

Email address: ben@benblohowiak.com