MULTIPOLAR INNER PRODUCT SPACES

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Abstract. A principle expressed by the associative property of addition constrains the structure of inner product spaces over fields such that objects embedded within those spaces may have certain properties (e.g., chirality). Fields may be generalized such that an addition-like operation (*consolidation*) is nonassociative and inner product spaces may be generalized over instances of those structures such that they have novel computational and geometric properties. Computability of length and angle in real numbers for vectors of such spaces makes possible practical applications such as cosine similarity analysis of data encoded via multipolar scales, e.g., data consisting of a spectrum having > 2 mutually inverse directions of extension. In this document, germane algebraic structures are defined, classes of number systems constructed, and preliminary geometric findings discussed.

Contents

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1. Introduction and Motivation

In 1827, Möbius provided a beginning sketch of what has become higher-dimensional vector analysis, writing that an object chiral in a space of dimension n may be mapped to its mirror image via a rotation of π radians through a space of dimension $n+1$.[1] Increasing the dimension of $\mathbb{R}^n$ by adding an orthogonal axis to that space doubles the quantity of orthants it contains as per the expression 2^n (e.g., four orthants or quadrants in the case of $\mathbb{R}^2$). The constant 2 in the expression 2^n may be attributed to the quantity of signs contained by the real number system ($\mathbb{R}$). In the case of the reals, the quantity of signs may be constrained by an axiomatic assumption of total order (comparability between all elements) or the associativity of addition; one implies the other. Generalization of Euclidean spaces to metric spaces containing p^n orthants such that $p > 2$ may require defining them over number systems in which addition is generalized such that its associative property is suspended and order relations become partial.

The following document supports the finding that an object chiral in a Euclidean space of dimension n may be mapped to its mirror image via translation by embedding it in a space of p^n orthants such that $p > 2$. In such a scenario, increase in the space's dimension is not required to render the embedded object achiral. Rather, increase in the quantity of mutually inverse coaxial directions (i.e., axial poles separated by an angle of π radians) may suffice. In this sense, an object's chirality is not only determined by its asymmetry or the dimension of the space in which it is embedded but also by the quantity of that space's axial poles.

Constructions of number systems with generalized addition over which multiplication distributes appear below, as do constructions of generalized inner product operations that permit computation of vector lengths and angles in real numbers for spaces over number systems with more than two signs. In addition to the novel geometric properties of such spaces, inner product computability makes possible cosine similarity analysis, such as that which may be used in interpretation of data collected via multipolar scales (i.e., data consisting of a spectrum having > 2 mutually inverse directions of extension).

2. Algebraic Structures

Algebraic structures with addition tend to be defined such that addition is associative. Noteworthy exceptions to this include neofields[4], left neofields[3], as well as the nonassociative number theory of Evans[2]. The addition operation of some neofields may neither commute nor associate and in those ways may differ from the addition of ordinary arithmetic or vector addition. In the case of Evans' system, Williams found that its properties include addition that neither commutes nor associates and has "no analogs in ordinary arithmetic."[5]

Defined below, two structures, *confields* and *comfields*, are defined such that they constitute superclasses of fields and correspond to number systems constructed in this document.

A multipolar number system can contain an embedding of the real line and a complex multipolar number system can contain an embedding of the complex plane. In general, a multipolar number system of p signs may contain an embedding of the real line iff p is double an odd integer value. More formally, $\mathbb{R} \in M_{C(p)} \iff p = 2n | n \bmod 2 = 1$. Similarly, a complex multipolar number system of p signs may contain an embedding of the complex plane iff p is double an odd integer value. More formally, $\mathbb{R} \in \mathbb{C} \in Y_{C(p)} \iff p = 2n | n \bmod 2 = 1$. For $x, y \in Y_{C(p)}, x = i \wedge y = -i \iff x \neq y \wedge p = 2n | n \bmod 2 = 1 \wedge x^4 = y^4 = 1$.

Generalizations of vector and inner product spaces are also axiomatized such that they may be defined over *confields*, of which *multipolar number systems* $(M_{G(p)})$ are instances, and their superclass *comfields*, of which *complex multipolar number systems* $(Y_{C(p)})$ are instances.

2.1. Confield Axioms.

Let a confield M be a set with two operations: consolidation and multiplication. The result of consolidating a and b is called the "consolid" or "c-sum" of a and b and may be denoted $a + b$ or $a \ddot{+} b$. The result of multiplying a and b is called the "product" of a and b, and may be denoted ab or $a \bullet b$. These operations are required to satisfy seven properties, referred to as "confield axioms." In these axioms, a, b, c, and d are arbitrary elements of the confield M.

1. Associativity of multiplication: $a \bullet (b \bullet c) = (a \bullet b) \bullet c$
2. Commutativity of consolidation and multiplication: $a \ddot{+} b = b \ddot{+} a$ and $a \bullet b = b \bullet a$
3. Consolidative identity: an element exists in M, 0, such that $a \ddot{+} 0 = a$.
4. Multiplicative identity: an element exists in M, 1, such that $a \bullet 1 = a$.
5. Consolidative inverses: For every $a \in M$ there exists one or more elements known as a consolidative inverse $b \in M$ such that $a \ddot{+} b = 0$.
6. Multiplicative inverses: for every $a \neq 0 \in M$, there exists an element in M that may be denoted by a^{-1} and called "the multiplicative inverse" of a, such that $a \bullet a^{-1} = 1$.
7. Left and right distributivity of multiplication over consolidation: $a \bullet (b \ddot{+} c) = (a \bullet b) \ddot{+} (a \bullet c) = (b \ddot{+} c) \bullet a$

While consolidation is not bound by axiom to associate, it may. One may construct a class of confield number systems (p-partite multipolars $(M_{G(p)})$) such that a given nonzero number may have $p - 1$ consolidative inverses. As $M_{C(2)}$ is the same as $\mathbb{R}$ up to an isomorphism it may exhibit extensional equality between confield consolidation and field addition.

2.2. **Comfield Axioms.** Let a comfield Y be a set with two operations: consolidation and multiplication. The result of consolidating a and b is called the "consolid" or "c-sum" of a and b and may be denoted $a + b$ or $a\check{+}b$. The result of multiplying a and b is called the "product" of a and b, and may be denoted ab or $a \bullet b$. These operations are required to satisfy seven properties, referred to as "comfield axioms." In these axioms, a, b, c, and d are arbitrary elements of the comfield Y.

1. Commutativity of consolidation and multiplication: $a\ddot{+}b = b\ddot{+}a$ and $a\bullet b = b\bullet a$
2. Consolidative identity: an element exists in Y, 0, such that $a\ddot{+}0 = a$.
3. Multiplicative identity: an element exists in Y, 1, such that $a \bullet 1 = a$.
4. Consolidative inverses: For every $a \in Y$ there exists one or more elements known as a consolidative inverse $b \in Y$ such that $a\ddot{+}b = 0$.
5. Multiplicative inverses: for every $a \neq 0 \in Y$, there exists an element in Y that may be denoted by a^{-1} and called "a multiplicative inverse" of a, such that $a \bullet a^{-1} = 1$.
6. Left and right distributivity of multiplication over consolidation: $a \bullet (b\ddot{+}c) = (a \bullet b)\ddot{+}(a \bullet c) = (b\ddot{+}c) \bullet a$

While consolidation is not bound by axiom to associate, it may. Similarly, while comfield multiplication is not bound by axiom to associate, it may. A nonzero element contained by a comfield does not necessarily have a unique multiplicative inverse (i.e., it may have more than one inverse). One may construct a class of comfield number systems (p-partite complex multipolars ($Y_{C(p)}$)) such that a given nonzero number may have $(p-1)^2$ consolidative inverses. As $Y_{C(2)}$ is the same as $\mathbb{C}$ up to an isomorphism it may exhibit extensional equality between comfield consolidation and field addition.

2.3. **Convector Spaces: Nonassociative Vector Consolidation.** Axioms regarding vector addition that define vector spaces may be suspended and replaced with those pertaining to vector consolidation, resulting in *convector spaces*. Multiplication by a comfield scalar may subsequently distribute over such consolidation. One may define vector consolidation for a convector space over a comfield such that vector spaces over fields are a subset of convector spaces over comfields, as is done below.

A convector space over a comfield Y may be defined as a set V with two operations, vector consolidation and scalar multiplication. The result of consolidating any two vectors v and w is called the "consolid" or "c-sum" of v and w and may be denoted $v\ddot{+}w$. The result of scalar multiplication of comfield element a and vector v is called the "product" of a and v, and may be denoted av or $a \bullet v$. These operations are required to satisfy seven axioms of convector spaces. In these axioms, u, v, and w are arbitrary vectors and a, b are scalars of the comfield Y.

1. Commutativity of consolidation: $u\ddot{+}v = v\ddot{+}u$
2. Identity element of consolidation: There exists an element $0 \in V$, called the zero vector, such that $v\ddot{+}0 = v$ for all $v \in V$.
3. Inverse elements of consolidation: For every vector $v \in V$, there exists one or more elements $u \in V$ called a consolidative inverse of v such that $v\ddot{+}u = 0$.
4. Compatibility of scalar multiplication with comfield multiplication: $v(ab) = (ab)v$

5. Identity element of scalar multiplication: $1v = v$, where 1 denotes the multiplicative identity in Y.

6. Distributivity of scalar multiplication with respect to vector consolidation: $a(u \ddot{+} v) = au \ddot{+} av$

7. Distributivity of scalar multiplication with respect to comfield consolidation: $(a \ddot{+} b)v = av \ddot{+} bv$

While comfield multiplication is not necessarily associative, confield multiplication is associative and so convector spaces over confields exhibit compatibility of scalar multiplication and confield multiplication such that $a(bv) = (ab)v$.

2.4. **Multipolar Inner Product Spaces.** Suspending the requirement of linearity in the first argument from canonical definitions of inner product spaces permits their generalization to *multipolar inner product spaces*. To be clear, such spaces may exhibit properties of additivity and homogeneity, though they are not necessarily constrained to do so. A multipolar inner product space is a convector space V over a comfield Y with a multipolar inner product operation (such as $\boxdot$ or $\ddot{\boxdot}$ defined below) that maps two vectors to a scalar in $\mathbb{R}$ such that it has the following properties:

1. Positivity: $v \ddot{\boxdot} v \geq 0$ for all $v \in V$
2. Definiteness: $v \ddot{\boxdot} v = 0 \iff \underline{v = 0}$
3. Conjugate symmetry $u \ddot{\boxdot} v = v \ddot{\boxdot} u$ for all $u, v \in V$

With the length of a vector defined as equal to the square root of its multipolar inner product with itself ($\|v\| = \sqrt{v \ddot{\boxdot} v}$), one may compute the angle between two vectors using the equation $\angle(u, v) = \arccos \frac{u \ddot{\boxdot} v}{\|u\|\|v\|}$ as they might for an inner product space proper. One may use this computation in applications such as cosine similarity analysis.

3. Definition of p-partite Multipolar Sets $M_{G(p)}$

For {

$$(x_{ai}, x_{a(i+1)}, x_{a(i+2)}, ..., x_{ap}) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times ... \times \mathbb{R}_{\geq 0}$$

and

$$(x_{bi}, x_{b(i+1)}, x_{b(i+2)}, ..., x_{bp}) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times ... \times \mathbb{R}_{\geq 0}$$

,

$$\{x_{ai}, x_{a(i+1)}, x_{a(i+2)}, ..., x_{ap}\} = A$$

and

$$\{x_{bi}, x_{b(i+1)}, x_{b(i+2)}, ..., x_{bp}\} = B$$

such that $|A_{>0}| \leq 2$ and $|B_{>0}| \leq 2$;

$$x'_{zi} = x_{zi} - x_{z(i+1)} - x_{z(i+2)} - ... - x_{zp}$$
$$x'_{z(i+1)} = x_{z(i+1)} - x_{zi} - x_{z(i+2)} - ... - x_{zp}$$
$$x'_{z(i+2)} = x_{z(i+2)} - x_{zi} - x_{z(i+1)} - ... - x_{zp}$$
$$x'_{z(...)} = x_{z(...)} - x_{zi} - x_{z(i+1)} - ... - x_{zp}$$
$$x'_{zp} = x_{zp} - x_{zi} - x_{z(i+1)} - x_{z(i+2)} - ... - x_{z(p-1)}$$

;

$$x''_{zi} = \operatorname{sgn}(1 + \operatorname{sgn}(x'_{zi}))x'_{zi}$$

},

$$(x_{ai}, x_{a(i+1)}, x_{a(i+2)}, ..., x_{ap}) \equiv (x_{bi}, x_{b(i+1)}, x_{b(i+2)}, ..., x_{bp})$$
$$\iff$$
$$(x''_{ai}, x''_{a(i+1)}, x''_{a(i+2)}, ..., x''_{ap}) = (x''_{bi}, x''_{b(i+1)}, x''_{b(i+2)}, ..., x''_{bp})$$
$$M_{G(p)} = (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times ... \times \mathbb{R}_{\geq 0})/\equiv \text{ such that } [(n, 0, ..., 0)] = n \in \mathbb{R}$$

4. Order Relations on p-partite Multipolar Sets $M_{G(p)}$

Below are four examples of how one may introduce p unique strict weak order relations on a p-partite multipolar set. A geometric intuition regarding these relations is that each references a unique direction of unbounded magnitude in a space of dimension one, i.e. each relation indicates relative distance from one of p maximal extensions (poles) of a p-partite multipolar number line. Under such an order relation, for each unique absolute value in $M_{G(p>2)}$ there are $p-1$ elements incomparable under that order relation. An example using a quadripartite multipolar system: $(0, 1, 0, 0)$, denoted sequitive one or S1, $(0, 0, 1, 0)$, denoted trinitive one or T1, and $(0, 0, 0, 1)$, denoted quadritive one or Q1, are incomparable with each other under the order relationship *less positive-ward than*, i.e., all indicated elements may be equidistant from a given coaxial positive vector.

4.1. Example Order Relation: Less Positive-ward Than (<Pos).

$$[(x_{ai}, x_{a(i+1)}, x_{a(i+2)}, ..., x_{ap})] < \text{Pos}[(x_{bi}, x_{b(i+1)}, x_{b(i+2)}, ..., x_{bp})]$$
$$\iff$$
$$x''_{ai} - (\text{sgn}((-(\text{sgn}(x'_{ai}))) + 1)(|(((|(|(|x_{a(i+1)} - x_{a(i+2)}|) - ...|) - x_{ap}|) - x_{ai})|)$$
$$<$$
$$x''_{bi} - (\text{sgn}((-(\text{sgn}(x'_{bi}))) + 1)(|(((|(|(|x_{b(i+1)} - x_{b(i+2)}|) - ...|) - x_{bp}|) - x_{bi})|)$$

4.2. Example Order Relation: Less Sequitive-ward Than (<Seq).

$$[(x_{ai}, x_{a(i+1)}, x_{a(i+2)}, ..., x_{ap})] < \text{Seq}[(x_{bi}, x_{b(i+1)}, x_{b(i+2)}, ..., x_{bp})]$$
$$\iff$$
$$x''_{ai+1} - (\text{sgn}((-(\text{sgn}(x'_{ai+1}))) + 1)(|(((|(|(|x_{a(i)} - x_{a(i+2)}|) - ...|) - x_{ap}|) - x_{ai+1})|)$$
$$<$$
$$x''_{bi+1} - (\text{sgn}((-(\text{sgn}(x'_{bi+1}))) + 1)(|(((|(|(|x_{b(i)} - x_{b(i+2)}|) - ...|) - x_{bp}|) - x_{bi+1})|)$$

4.3. Example Order Relation: Less Trinitive-ward Than (<Trin).

$$[(x_{ai}, x_{a(i+1)}, x_{a(i+2)}, ..., x_{ap})] < \text{Trin}[(x_{bi}, x_{b(i+1)}, x_{b(i+2)}, ..., x_{bp})]$$
$$\iff$$
$$x''_{ai+2} - (\text{sgn}((-(\text{sgn}(x'_{ai+2}))) + 1)(|(((|(|(|x_{a(i)} - x_{a(i+1)}|) - ...|) - x_{ap}|) - x_{ai+2})|)$$
$$<$$
$$x''_{bi+2} - (\text{sgn}((-(\text{sgn}(x'_{bi+2}))) + 1)(|(((|(|(|x_{b(i)} - x_{b(i+1)}|) - ...|) - x_{bp}|) - x_{bi+2})|)$$

4.4. Example Order Relation: Less p-itive-ward Than ($< p$).

$$[(x_{ai}, x_{a(i+1)}, x_{a(i+2)}, ..., x_{ap})] < p[(x_{bi}, x_{b(i+1)}, x_{b(i+2)}, ..., x_{bp})]$$
$$\iff$$
$$x''_{ap} - (\text{sgn}((-(\text{sgn}(x'_{ap}))) + 1)(|(((|(|(|x_{a(i)} - x_{a(i+1)}|) - x_{ai+2}|) - ...|) - x_{ap})|)$$
$$<$$
$$x''_{bp} - (\text{sgn}((-(\text{sgn}(x'_{bp}))) + 1)(|(((|(|(|x_{b(i)} - x_{b(i+1)}|) - x_{bi+2}|) - ...|) - x_{bp})|)$$

5. p-partite Multipolar Unary Operations

5.1. Unary Operation: Multipolar Absolute Value ($\dot{|}x\dot{|}$). The absolute value of a multipolar number is its value irrespective of its sign. $\dot{|}x\dot{|}$: $M_{G(p)} \mapsto \mathbb{R}_{\geq 0}$

$$\dot{|}[(x_{zi}, ..., x_{zp})]\dot{|} = [(x''_{zi} + x''_{zi+1} + ... + x''_{zp}, 0, ..., 0)]$$

5.2. Unary Operation: Multipolar Sign ($\mathrm{msgn}(x)$). The sign of a multipolar number is either a positive unit or one of its $p-1$ consolidative inverses. $\mathrm{msgn}(x)$: $M_{G(p)} \mapsto M_{G(p)}$

$$\mathrm{msgn}([(x_{zi}, ..., x_{zp})]) = [(\mathrm{sgn}(x''_{zi}), \mathrm{sgn}(x''_{z(i+1)}), ..., \mathrm{sgn}(x''_{zp}))]$$

6. p-partite Multipolar Binary Operations

6.1. Binary Operation: Multipolar Consolidation ($\ddot{+}$). Consolidation generalizes field addition such that it does not necessarily associate. Consolidation commutes and its identity element is zero. $\ddot{+}$: $M_{G(p)} \times M_{G(p)} \mapsto M_{G(p)}$

Consolidation ($\ddot{+}$) may be defined:

$$[(x_{ai}, ..., x_{ap})]\ddot{+}[(x_{bi}, ..., x_{bp})] = [((x''_{ai} + x''_{bi}), (x''_{ai+1} + x''_{bi+1}), ..., (x''_{ap} + x''_{bp}))]$$

A given element may have a non-unique inverse under consolidation and subsequently consolidation may not exhibit properties consistent with associativity such as outputs bound by the cancellation law. Extensional equality between addition and consolidation may be shown with $M_{C(2)}$.

6.2. Binary Operation: Multipolar Multiplication ($\bullet$). The p-partite multipolars are Abelian groups under multiplication that distributes over consolidation. Sign multiplication tables for multipolar systems may be the same as the Cayley tables of finite Abelian groups of order p (e.g., K_4, C_8, etc.) up to an isomorphism. One may construct as many multipolar systems of a given p as there are finite Abelian groups of order p; different systems of the same p (sign cardinality) may be distinguished by the finite Abelian group used to construct the multiplication operation for the system (e.g., $M_{K(4)}$ and $M_{C(4)}$). $\bullet$: $M_{G(p)} \times M_{G(p)} \mapsto M_{G(p)}$

Suppose $\{a, b, c, d\} \in M_{G(p)}$;

$$\mathrm{msgn}(a) \bullet \mathrm{msgn}(b) = \mathrm{msgn}(c) = \mathrm{msgn}(a) \bullet \mathrm{msgn}(d) \iff \mathrm{msgn}(b) = \mathrm{msgn}(d)$$

i.e., in a given system $M_{G(p)}$, a given product sign and a given factor sign imply one and only one sign for the other multiplied factor.

Multiplication ($\bullet$) for tripartite multipolars ($M_{C(3)}$): $[(x_{a1}, x_{a2}, x_{a3})]\bullet[(x_{b1}, x_{b2}, x_{b3})] = [((x''_{a1} \times x''_{b1}) + (x''_{a2} \times x''_{b3}) + (x''_{a3} \times x''_{b2})), ((x''_{a1} \times x''_{b2}) + (x''_{a2} \times x''_{b1}) + (x''_{a3} \times x''_{b3})), ((x''_{a1} \times x''_{b3}) + (x''_{a2} \times x''_{b2}) + (x''_{a3} \times x''_{b1}))]$

Multiplication ($\bullet$) for quadripartite multipolars ($M_{C(4)}$): $[(x_{a1}, x_{a2}, x_{a3}, x_{a4})] \bullet [(x_{b1}, x_{b2}, x_{b3}, x_{b4})] = [((x''_{a1} \times x''_{b1}) + (x''_{a2} \times x''_{b4}) + (x''_{a3} \times x''_{b3}) + (x''_{a4} \times x''_{b2})), ((x''_{a1} \times x''_{b2}) + (x''_{a2} \times x''_{b1}) + (x''_{a3} \times x''_{b4}) + (x''_{a4} \times x''_{b3})), ((x''_{a1} \times x''_{b3}) + (x''_{a2} \times x''_{b2}) + (x''_{a3} \times x''_{b1}) + (x''_{a4} \times x''_{b4})), ((x''_{a1} \times x''_{b4}) + (x''_{a2} \times x''_{b3}) + (x''_{a3} \times x''_{b2}) + (x''_{a4} \times x''_{b1}))]$

Multiplication ($\bullet$) for quadripartite multipolars ($M_{K(4)}$): $[(x_{a1}, x_{a2}, x_{a3}, x_{a4})] \bullet [(x_{b1}, x_{b2}, x_{b3}, x_{b4})] = [((x''_{a1} \times x''_{b1}) + (x''_{a2} \times x''_{b2}) + (x''_{a3} \times x''_{b3}) + (x''_{a4} \times x''_{b4})), ((x''_{a1} \times x''_{b2}) + (x''_{a2} \times x''_{b1}) + (x''_{a3} \times x''_{b4}) + (x''_{a4} \times x''_{b3})), ((x''_{a1} \times x''_{b3}) + (x''_{a2} \times x''_{b4}) + (x''_{a3} \times x''_{b1}) + (x''_{a4} \times x''_{b2})), ((x''_{a1} \times x''_{b4}) + (x''_{a2} \times x''_{b3}) + (x''_{a3} \times x''_{b2}) + (x''_{a4} \times x''_{b1}))]$

7. Definition of Complex Multipolar Sets $Y_{C(p)}$

If p mod $2 = 0$, for $\{ (a, b) \in M_{C(p)} \times M_{C(p)}$ and $(c, d) \in M_{C(p)} \times M_{C(p)} \}$,

$$(a, b) \equiv (c, d) \iff a = c \text{ and } b = d$$

$$Y_{C(p)} = (M_{C(p)} \times M_{C(p)})/\equiv \text{ such that } (n, 0) = n \in M_{C(p)}$$

Multipolar number systems may each be extended by an imaginary-like element if they are constructed such that their sign multiplication tables are the same up to isopmorphism as the Cayley tables of cyclic finite groups (C_p) containing an even number of elements (p mod $2 = 0$). In such number systems extended by an imaginary-like element, the coefficients of that element are denoted b and d in the construction above. In general, the element itself (i.e., if $b \vee d = 1$) may be denoted by s such that it is its system's $(2p)$th root of unity (i.e., $s^{2p} = 1$, $s^{(p-(p-2))} =$ sequitive one (S1), etc.). In this sense, for $s \in Y_{C(2)}$ and $i \in \mathbb{C}$, $s = i$.

In general, a complex multipolar number system of p signs may contain an embedding of the complex plane iff p is double an odd integer value. More formally, $\mathbb{R} \in \mathbb{C} \in Y_{C(p)} \iff p = 2n | n \text{mod} 2 = 1$. For $x, y \in Y_{C(p)}, x = i \wedge y = -i \iff x \neq y \wedge p = 2n | n \text{mod} 2 = 1 \wedge x^4 = y^4 = 1$.

Unary and binary operations for complex multipolars are defined in the sections below.

8. Complex Multipolar Unary Operations & Functions

8.1. Complex Multipolar Absolute Value ($\ddot{|}x\ddot{|}$). The absolute value of a complex multipolar generalizes a Euclidean norm such that it is a nonnegative real value. $\ddot{|}x\ddot{|}$: $Y_{C(p)} \mapsto \mathbb{R}_{\geq 0}$

$$\ddot{|}(a, b)\ddot{|} = \sqrt{\dot{|}a\dot{|}^2 + \dot{|}b\dot{|}^2}$$

8.2. Complex Multipolar Sine ($\sin(x)$) and Cosine ($\cos(x)$). The multipolar sine and cosine of an angle measured in real radians x may be defined by a cyclic path of $p\pi$ radians around the origin as traced by exponentiation of an imaginary-like units. In the most general case, such an imaginary-like unit is signed positive such that there are $(p/2) - 1$ other unit(s) that define unique orthogonal paths; not all such paths trace an equal circumference (e.g., S$s \in M_{C(6)}$).

$\sin(x)$: $\mathbb{R}_{\geq 0} \mapsto Y_{C(p)}$, $\cos(x)$: $\mathbb{R}_{\geq 0} \mapsto Y_{C(p)}$

Supposing $x \in \mathbb{R}_{\geq 0}$,

$$(0, 1)^{(2x)/\pi} = s^{(2x)/\pi} = (\cos(x), \sin(x))$$

8.3. **Real-like Part of a Complex Multipolar Number ($\mathrm{Rel}(a,b)$).** The real-like part of a complex multipolar is a multipolar value. $\mathrm{Rel}(x)$: $Y_{C(p)} \mapsto M_{C(p)}$

$$\mathrm{Rel}(a,b) = a$$

8.4. **Imaginary-like Part of a Complex Multipolar Number ($\mathrm{Iml}(a,b)$).** The imaginary-like part of a complex multipolar is a multipolar value. $\mathrm{Iml}(x)$: $Y_{C(p)} \mapsto M_{C(p)}$

$$\mathrm{Iml}(a,b) = b$$

8.5. **Complex Multipolar Conjugate ($\overline{x}$).** If a complex multipolar is consolidated with its conjugate, its imaginary-like part is annihilated and its real-like part is doubled. $\overline{x}$: $Y_{C(p)} \mapsto Y_{C(p)}$

Supposing $a,b,c \in M_{C(p)}$ such that $c = ((0,1)^p)b = (s^p)b$,

$$\overline{(a,b)} = (a,c)$$

9. Complex Multipolar Binary Operations & Functions

9.1. **Complex Multipolar Consolidation ($\breve{+}$).** Consolidation generalizes field addition such that it does not necessarily associate. Consolidation commutes and its identity element is zero. $\breve{+}$: $Y_{C(p)} \times Y_{C(p)} \mapsto Y_{C(p)}$

Complex Multipolar Consolidation ($\breve{+}$) may be defined:

$$(a,b)\breve{+}(c,d) = (a\ddot{+}c, b\ddot{+}d)$$

A given element may have a non-unique inverse under consolidaiton and subsequently consolidation may not exhibit properties consistent with associativity such as outputs bound by the cancellation law. Extensional equality between addition and consolidation may be shown with $Y_{C(2)}$.

9.2. **Complex Multipolar Multiplication ($\breve{\bullet}$).** Multiplication of complex multipolars distributes over consolidation. Nonzero complex multipolars have at least one multiplicative inverse though such inverses are not necessarily unique and thus outputs of multiplication may not be bound by the cancellation law. $\breve{\bullet}$: $Y_{C(p)} \times Y_{C(p)} \mapsto Y_{C(p)}$

Multiplication ($\breve{\bullet}$) for complex multipolars ($Y_{C(p)}$):

$$(a,b)\breve{\bullet}(c,d) = (ac\ddot{+}(\mathrm{S1}(bd)), ad\ddot{+}bc)$$

10. Convector Spaces and Multipolar Inner Products

10.1. **Vector Equivalence.** Vectors may be expressed as columns with their equality defined as below. Suppose that $u,v \in M^n_{G(p)}$ or $\in Y^n_{C(p)}$. Then $u = v$ if $[u]_i = [v]_i$ and $1 \leq i \leq n$.

10.2. **Binary Operation: Vector Consolidation.** Vectors expressed as columns may be consolidated. $\ddot{+}$: $V \times V \mapsto V$

Supposing that $u,v \in M^n_{G(p)}$ or $\in Y^n_{C(p)}$ and $1 \leq i \leq n$, the consolid of u and v is the vector $u\ddot{+}v$ as indicated: $[u\ddot{+}v]_i = [u]_i\ddot{+}[v]_i$

10.3. **Binary Operation: Scalar Multiplication.** Vectors expressed as columns may be multiplied by scalars. $M_{G(p)} \times V \mapsto V$ and $Y_{C(p)} \times V \mapsto V$

Supposing $u \in M^n_{G(p)}$ and $\alpha \in M_{G(p)}$ or $u \in Y^n_{C(p)}$ and $\alpha \in Y_{C(p)}$, then the scalar multiple of u by α is the vector αu defined by $[\alpha u]_i = \alpha[u]_i$ and $1 \leq i \leq n$.

10.4. **Binary Function: Multipolar-to-Real Product ($\mathrm{mrp}(x, y)$).** The multipolar-to-real product function maps the magnitude of the product of two multipolar numbers to a real number whose sign depends on whether the signs of the multiplied numbers are identical. $\mathrm{mrp}(x, y)$: $M_{G(p)} \times M_{G(p)} \mapsto \mathbb{R}$

Supposing $\{x, y\} \in M_{G(p)}$,

$$\mathrm{mrp}(x,y) = \begin{cases} \dot{|}x\dot{|}\dot{|}y\dot{|} \iff & |\,\mathrm{msgn}(\mathrm{msgn}(x) \ddot{+} \mathrm{msgn}(y))| = 1 \\ -(\dot{|}x\dot{|}\dot{|}y\dot{|}) \iff & |\,\mathrm{msgn}(\mathrm{msgn}(x) \ddot{+} \mathrm{msgn}(y))| = 0 \end{cases}$$

10.5. **Binary Operation: Multipolar Inner Product ($u \boxdot v$).** Multipolar inner product allows computation of vector length and angle between two vectors in $M^n_{G(p)}$. $\boxdot$: $V \times V \mapsto \mathbb{R}$

Supposing $\{(u_i, u_{i+1}, ..., u_n), (v_i, v_{i+1}, ..., v_n)\} \in M^n_{G(p)}$, multipolar inner product of two vectors $u = (u_i, u_{(i+1)}, ..., u_n)$ and $v = (v_i, v_{(i+1)}, ..., v_n)$ is defined as:

$$u \boxdot v = \sum_{i=1}^{n} \mathrm{mrp}(u_i v_i) = \mathrm{mrp}(u_1 v_1) + \mathrm{mrp}(u_2 v_2) + ... + \mathrm{mrp}(u_n v_n)$$

As indicated above, $\|v\| = \sqrt{v \boxdot v}$ such that one may compute the angle between two vectors using the equation $\angle(u, v) = \arccos \frac{u \boxdot v}{\|u\|\|v\|}$.

10.6. **Binary Operation: Complex Multipolar Inner Product ($u \ddot{\boxdot} v$).** Complex multipolar inner product allows computation of vector length and angle between two vectors in $Y^n_{C(p)}$. $\ddot{\boxdot}$: $V \times V \mapsto \mathbb{R}$

Supposing $\{(u_i, u_{i+1}, ..., u_n), (v_i, v_{i+1}, ..., v_n)\} \in Y^n_{C(p)}$ and

$$u'_i = \mathrm{Rel}(u_i), u''_i = \mathrm{Iml}(u_i),$$
$$v'_i = \mathrm{Rel}(v_i), v''_i = \mathrm{Iml}(v_i),$$
$$u'_{...} = \mathrm{Rel}(u_{...}), u''_{...} = \mathrm{Iml}(u_{...}),$$
$$v'_{...} = \mathrm{Rel}(v_{...}), v''_{...} = \mathrm{Iml}(v_{...}),$$
$$u'_n = \mathrm{Rel}(u_n), u''_n = \mathrm{Iml}(u_n),$$
$$v'_n = \mathrm{Rel}(v_n), v''_n = \mathrm{Iml}(v_n),$$

the complex multipolar inner product of two vectors $u = (u_i, u_{(i+1)}, ..., u_n)$ and $v = (v_i, v_{(i+1)}, ..., v_n)$ is defined as:

$$u \ddot{\boxdot} v = \sum_{i=1}^{n} \mathrm{mrp}(u_i v_i) = \mathrm{mrp}(u'_1 v'_1) + \mathrm{mrp}(u''_1 v''_1) + ... + \mathrm{mrp}(u'_n v'_n) + \mathrm{mrp}(u''_n v''_n)$$

As indicated above, $\|v\| = \sqrt{v \ddot{\boxdot} v}$ such that one may compute the angle between two vectors using the equation $\angle(u, v) = \arccos \frac{u \ddot{\boxdot} v}{\|u\|\|v\|}$.

11. Geometric Properties of Multipolar Spaces

11.1. **Multipolar Axes.** On a given axis $M^1_{G(p)}$, there are p coaxial directions from the origin, or poles. Each pair of vectors in those mutually opposing p directions is π radians apart (e.g., (1) and (S1), (1) and (T1), (S1) and (T1), etc.). By computation of distances, no two coaxial vectors of unlike sign may have a vector of a third sign between them.

One may visualize a multipolar axis as a complete graph (mystic rose) with p nodes such that each node indicates the unique direction of a coordinate vector of unbounded magnitude and the midpoint of each graph edge indicates the location of the origin. In such a visualization, mapping the same vector to more than one indicator may permit one to intuitively apprehend that the angle between any two points of a multipolar continuum is π radians.

One may multiply a vector on a p-partite multipolar axis with one of $p-1$ nonpositive scalar units such that its product is mapped to one of its $p-1$ coaxial consolidative inverse vectors π radians away. For reasons discussed below, such consolidative inverse vectors may be referred to as *linear antipodes*. Any member of a set of linear antipodes may be transformed to any of the others in that set by such nonpositive unit scalar multiplication.

In general, a multipolar number system of p signs may contain an embedding of the real line iff p is double an odd integer value. More formally, $\mathbb{R} \in M_{C(p)} \iff p = 2n | n \text{mod} 2 = 1$.

11.2. **Multipolar Planes.**

11.2.1. *Orthogonality.* In a multipolar space $M^n_{G(p)}$, each of the n axes intersect at the origin and form right angles ($\pi/2$ radians) with the other $n-1$ axes. Linear functions expressed via point-slope or parametric form have p-poled outputs such that each pole is π radians apart from the other.

11.2.2. *Achirality.* One may embed in any of the p^n orthants of $M^n_{G(p)}$ an object congruent to any in $\mathbb{R}^n$. If such an embedded object is chiral in $\mathbb{R}^n$ and is contained by an orthant of $M^n_{G(p>2)}$, it may be shown to be achiral in $M^n_{G(p>2)}$. This may be demonstrated by reflecting the object across an axial hyperplane of the space and then translating the object and its reflection by the same distance and direction across the axial hyperplane into a third orthant. In this sense, chirality is not only a function of dimension or object asymmetry but also axial pole quantity.

For example, a chiral object in orthant $(+,+)$ and its reflection across the y-axis in $(\text{S},+)$ may be mapped to coincide in orthant $(\text{T},+)$ via a translation vector $(Tx, 0)$ such that x is of sufficient magnitude to carry the objects across the y-axis. Intuitively, the unification of such an object and its reflection may be thought of as being brought to coincide in a manner similar to how the slider of a zipper unites its rows of teeth.

11.2.3. *Unit Circles and Linear Functions: Antipodes.* In an orthant of a multipolar space, a given point B on a unit circle or $(n-1)$-sphere may have an origin-centered linear function that passes through it. Such a linear function may intersect the unit circle or $(n-1)$-sphere at $p-1$ other points π radians away from B, B's linear antipodes. The total quantity of antipodes of B, or points π radians away from B on the surface of a unit circle or $(n-1)$-sphere, equals $(p-1)^n$. There are

$(p-1)^n - p + 1$ antipodes of B that are not its linear antipodes. Such antipodes that do not lie on the linear function that passes through B may be referred to as B's *nonlinear antipodes*. While each antipode of B is π radians from B, pairs of those antipodes may be $\leq \pi$ radians from each other. Whereas consolidation of a pair of linear antipode vectors results in their annihilation, consolidation of a pair of nonlinear antipode vectors may result in their annihilation or in a vector of nonzero magnitude.

11.2.4. *Unit Circles: Cycles of Exponentiation.* In the plane of $\mathbb{C}^1$, scaling of a vector by i rotates that vector about the origin $\pi/2$ radians; scaling the vector (1) by $i^{2\theta/\pi}$ may rotate it about the origin by any angle θ and in this way it may traverse the path of the unit circle, a Hamiltonian cycle through all orthants of the plane.

In $Y^1_{C(p>2)}$, scaling of a vector by s rotates that vector about the origin of a complex multipolar plane $\pi/2$ radians. However, scaling the vector (1) by $s^{2\theta/\pi}$ may not rotate it about the origin such that it traverses a Hamiltonian cycle through all orthants of the plane, despite it being able to rotate the given vector continuously around the origin by any angle θ. Although the angle by which such a rotated vector may travel is unbounded, its path may not include all orthants of its plane.

In multipolar planes $M^2_{G(p>2)}$, unit circles centered at the origin do not necessarily constitute Hamiltonian cycles among the p^2 orthants of each plane if p mod $2 = 1$. If p mod $2 = 0$, there are > 2 orthant sequences that trace circles of unit radius (e.g., $M^2_{K(4)}$).

For example, $Y^1_{C_4}$ has 16 orthants and scaling the vector (1) by $s^{2\theta/\pi}$ for all θ may rotate it through eight of them such that one complete cycle of its path is 4π radians. (Upon having been rotated in this manner 2π radians through its plane, the vector in question is transformed into a consolidative inverse of its starting coordinate, similar to a spinor's transformation into its negative by a rotation of 2π radians.) Scaling the vector (1) by $(\mathrm{S}s)^{2\theta/\pi}$ for all θ may rotate it through the remaining eight orthants of the plane such that its cycle and that of $s^{2\theta/\pi}$ intersect only at unit length on each axis.

Generally, the quantity of distinct cycles defined by differently signed versor-like scalars required to rotate (1) through a subset of the orthants of a complex multipolar plane in the path of a unit circle is $p/2$. Given a complex multipolar plane, not all exponentiation cycles necessarily trace an equal circumference (e.g., $\mathrm{S}s \in M^1_{C(6)}$); in such cases, cycle length may vary by versor-like scalar sign (e.g., 2π radians for $\mathrm{S}s \in M^1_{C(6)}$ and 6π radians for $s \in M^1_{C(6)}$) and there may be orthants through which no exponentiation cycle passes, orthants that do not contain roots of unity.

A complex multipolar number system of p signs may contain an embedding of the complex plane iff p is double an odd integer value. More formally, $\mathbb{R} \in \mathbb{C} \in Y_{C(p)} \iff p = 2n | n \text{mod} 2 = 1$. For $x, y \in Y_{C(p)}, x = i \wedge y = -i \iff x \neq y \wedge p = 2n | n \text{mod} 2 = 1 \wedge x^4 = y^4 = 1$.

11.3. **Euler Characteristics.** In $M^3_{G(p)}$, one may compute the Euler characteristics of polyhedra such as 3-orthotopes and their duals. (See Table 1 below.)

TABLE 1. For Computation of Euler Characteristics in $M^3_{G(p)}$

Polyhedron	**V**	**F**	**E**
3-orthotope	p^3	$3p$	$\frac{(p^3)(3(p-1))}{2}$
3-o. dual/fusil	$3p$	p^3	$3p^2$

In $M^3_{G(p>2)}$, 3-orthotopes and their duals do not have equal Euler characteristics. In such spaces, 3-orthotopes have negative Euler characteristics whereas their duals have positive Euler characteristics > 2, which may warrant further exploration.

11.4. **Data Science Applications.** Applications for multipolar number systems and multipolar inner product spaces over those systems may include analysis of data captured via multipolar scale items, such as those used to capture data consisting of a spectrum having > 2 mutually inverse directions of extension. Item responses and/or their scale values may be analysed by orthant for qualitative classification and, with the length of a vector defined as equal to the square root of its multipolar inner product with itself ($\|v\| = \sqrt{v \boxdot v}$), one may compute the angle between two vectors using the equation $\angle(u, v) = \arccos \frac{u \boxdot v}{\|u\|\|v\|}$ as they might for an inner product space proper. So doing, one may have the ability to perform cosine similarity analysis in multipolar inner product spaces.

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